Coordinates

Engineering Mechanics: Dynamics

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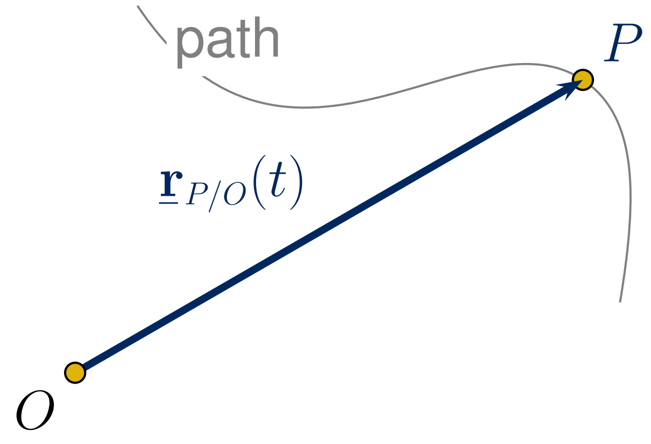
Dynamics describes the changing configuration of a system $\longrightarrow$ The configuration is defined in terms of position vectors

What is necessary to measure the position vector $\underline{\mathbf{r}}_{P/O}(t)$?

Basis: reference directions

Coordinates: measurable quantities (e.g. distance, angle) that are used specify the configuration of the system. Typically describe

- components with respect to specified directions, and/or
- the directions with respect to a reference basis (inertial space)



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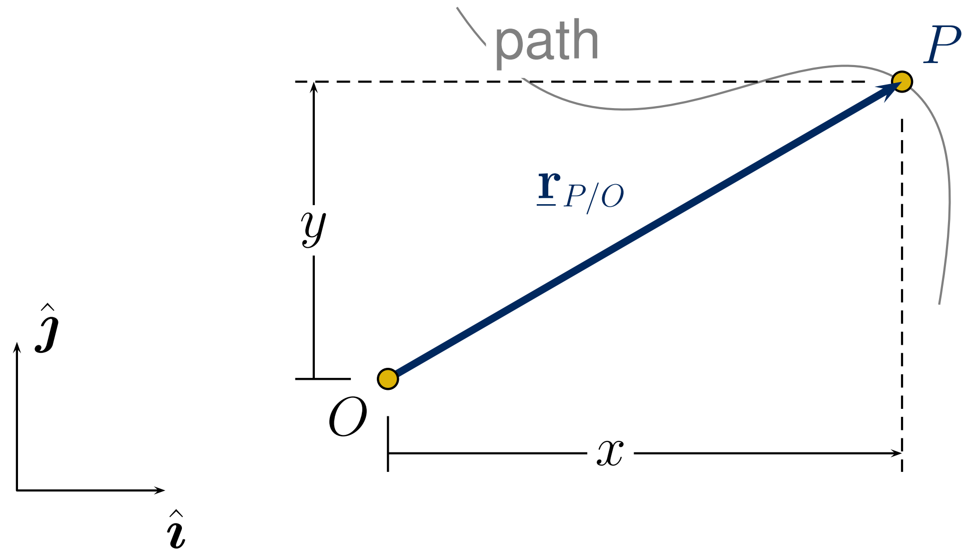
Cartesian Coordinates (René Descartes, 1596–1650)

The position vector can be described in terms of components of directions *fixed in the ground*

$$\underline{\mathbf{r}}_{P/O} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}},$$

$$\underline{\mathbf{v}}_P = \frac{d}{dt}\left(\underline{\mathbf{r}}_{P/O}\right) = \dot{x}\,\hat{\boldsymbol{\imath}} + \dot{y}\,\hat{\boldsymbol{\jmath}},$$

$$\underline{\mathbf{a}}_P = \frac{d^2}{dt^2}\left(\underline{\mathbf{r}}_{P/O}\right) = \frac{d}{dt}\left(\underline{\mathbf{v}}_P\right) = \ddot{x}\,\hat{\boldsymbol{\imath}} + \ddot{y}\,\hat{\boldsymbol{\jmath}}.$$



If $\ddot{x} = f(t)$

$$\int_0^t \left\{ \ddot{x}(\xi) = f(\xi) \right\} d\xi \quad \longrightarrow \quad \dot{x}(t) - \dot{x}(0) = \int_0^t f(\xi)\,d\xi,$$

$$\int_0^t \left\{ \dot{x}(\tau) = \dot{x}(0) + \int_0^\tau f(\xi)\,d\xi \right\} d\tau \quad \longrightarrow \quad x(t) - x(0) = \int_0^t \int_0^\tau f(\xi)\,d\xi\,d\tau + \dot{x}(0)\,t.$$

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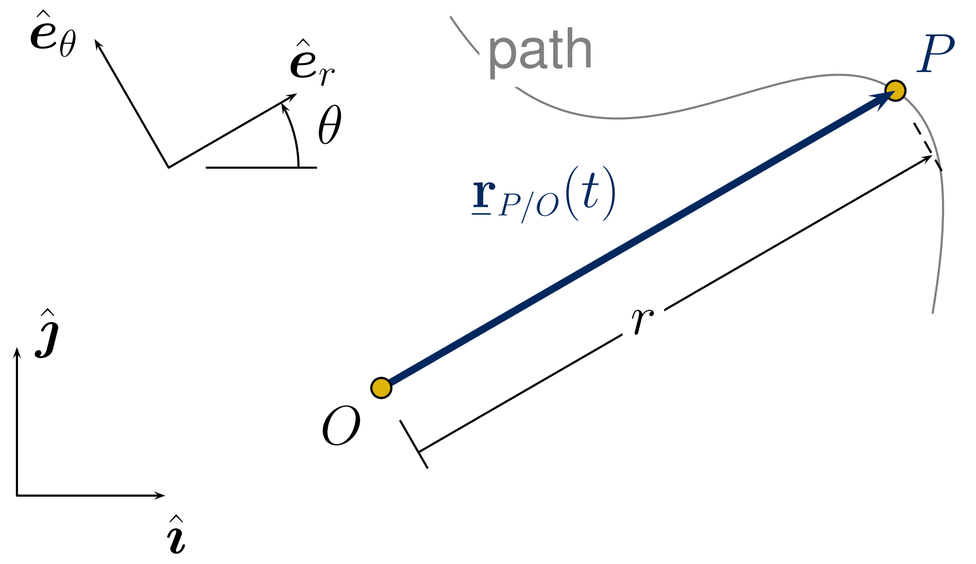
Polar Coordinates

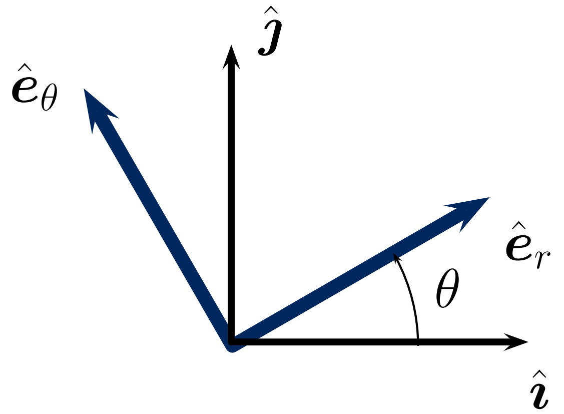
The position vector can be described in terms of the magnitude of the position and basis directions that *move with the position*, so that

$$\underline{\mathbf{r}}_{P/O} = r\,\hat{\boldsymbol{e}}_r.$$



The directions $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta)$ spin with angular velocity $\underline{\boldsymbol{\omega}} = \dot{\theta}\,\hat{\boldsymbol{k}}$ with respect to $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$, so that

$$\frac{d}{dt}\left(\hat{\boldsymbol{e}}_r\right) = \underline{\boldsymbol{\omega}} \times \hat{\boldsymbol{e}}_r = \left(\dot{\theta}\,\hat{\boldsymbol{k}}\right) \times \hat{\boldsymbol{e}}_r = \dot{\theta}\,\hat{\boldsymbol{e}}_\theta,$$
$$\frac{d}{dt}\left(\hat{\boldsymbol{e}}_\theta\right) = \underline{\boldsymbol{\omega}} \times \hat{\boldsymbol{e}}_\theta = \left(\dot{\theta}\,\hat{\boldsymbol{k}}\right) \times \hat{\boldsymbol{e}}_\theta = -\dot{\theta}\,\hat{\boldsymbol{e}}_r.$$



The polar directions $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta)$ can be related to the Cartesian directions $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$ as

$$\hat{\boldsymbol{e}}_r = \mathrm{C}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{S}_\theta\ \hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\imath}} = \mathrm{C}_\theta\ \hat{\boldsymbol{e}}_r - \mathrm{S}_\theta\ \hat{\boldsymbol{e}}_\theta,$$
$$\hat{\boldsymbol{e}}_\theta = -\,\mathrm{S}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{C}_\theta\ \hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\jmath}} = \mathrm{S}_\theta\ \hat{\boldsymbol{e}}_r + \mathrm{C}_\theta\ \hat{\boldsymbol{e}}_\theta$$

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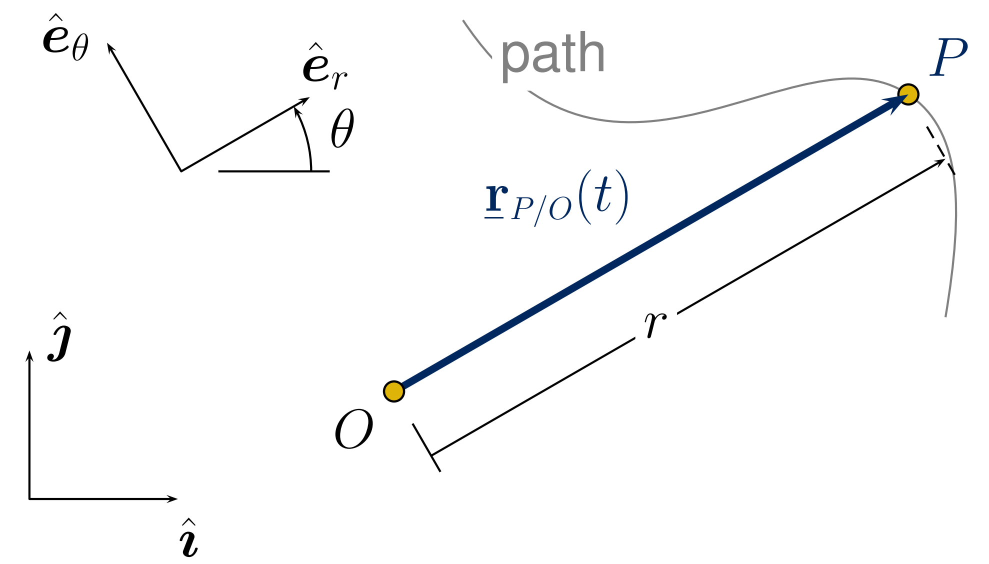
The position vector is $\underline{\mathbf{r}}_{P/O} = r\,\hat{\boldsymbol{e}}_r$, so that

$$\underline{\mathbf{v}}_P = \frac{d}{dt}\Big(\underline{\mathbf{r}}_{P/O}\Big) = \dot{r}\,\hat{\boldsymbol{e}}_r + r\,\frac{d}{dt}\Big(\hat{\boldsymbol{e}}_r\Big),$$
$$= \underbrace{\dot{r}\,\hat{\boldsymbol{e}}_r}_{\text{radial}} + \underbrace{r\,\dot{\theta}\,\hat{\boldsymbol{e}}_\theta}_{\text{transverse}},$$

and

$$\underline{\mathbf{a}}_P = \frac{d}{dt}\Big(\underline{\mathbf{v}}_P\Big) = \frac{d}{dt}\Big(\dot{r}\Big)\,\hat{\boldsymbol{e}}_r + \dot{r}\,\frac{d}{dt}\Big(\hat{\boldsymbol{e}}_r\Big)$$
$$+ \frac{d}{dt}\Big(r\,\dot{\theta}\Big)\,\hat{\boldsymbol{e}}_\theta + (r\,\dot{\theta})\,\frac{d}{dt}\Big(\hat{\boldsymbol{e}}_\theta\Big),$$
$$= \underbrace{\Big(\ddot{r} - r\,\dot{\theta}^2\Big)\,\hat{\boldsymbol{e}}_r}_{\text{radial}} + \underbrace{\Big(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\Big)\,\hat{\boldsymbol{e}}_\theta}_{\text{transverse}}.$$



The speed can be written as

$$v = \|\underline{\mathbf{v}}_P\| = \sqrt{\underline{\mathbf{v}}_P \cdot \underline{\mathbf{v}}_P},$$
$$= \sqrt{\dot{x}^2 + \dot{y}^2},$$
$$= \sqrt{\dot{r}^2 + \Big(r\,\dot{\theta}\Big)^2}.$$

The components of velocity and acceleration in the $\hat{e}_r$ and $\hat{e}_\theta$ directions are defined as the *radial* and *transverse* components.

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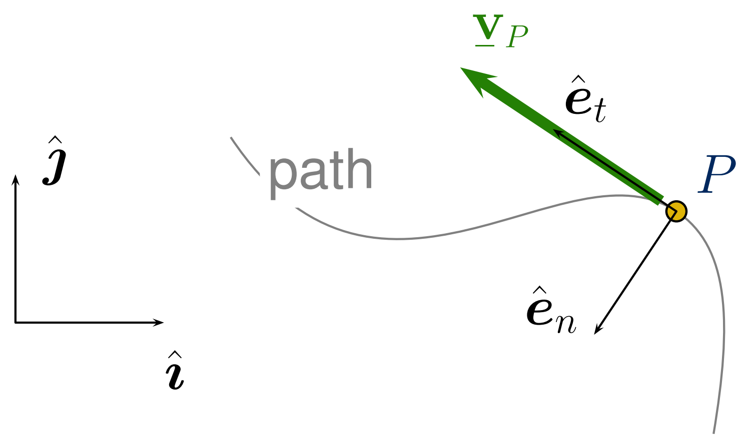
Normal and Tangential Coordinates

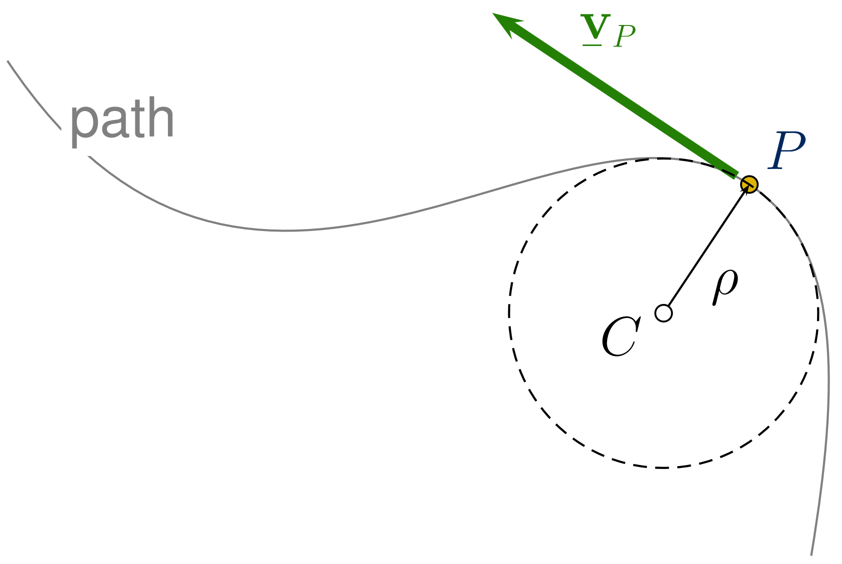
The velocity vector can be described in terms of its magnitude v and direction $\hat{\boldsymbol{e}}_t$, as

$$\underline{\mathbf{v}}_P = v\,\hat{\boldsymbol{e}}_t$$

The directions $(\hat{\boldsymbol{e}}_t, \hat{\boldsymbol{e}}_n)$ move with the velocity, and the acceleration is

$$\underline{\mathbf{a}}_P = \dot{v}\,\hat{\boldsymbol{e}}_t + \frac{v^2}{\rho}\,\hat{\boldsymbol{e}}_n.$$





ρ is defined as the *radius of curvature* while $\hat{\boldsymbol{e}}_n$ is directed toward the *center of curvature* C.

- For a circular path the radius of curvature is identical to the radius of the path, so that $\rho = r$
- If the path is written as $y = f(x)$, then

$$\rho = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}.$$

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