Particle Dynamics

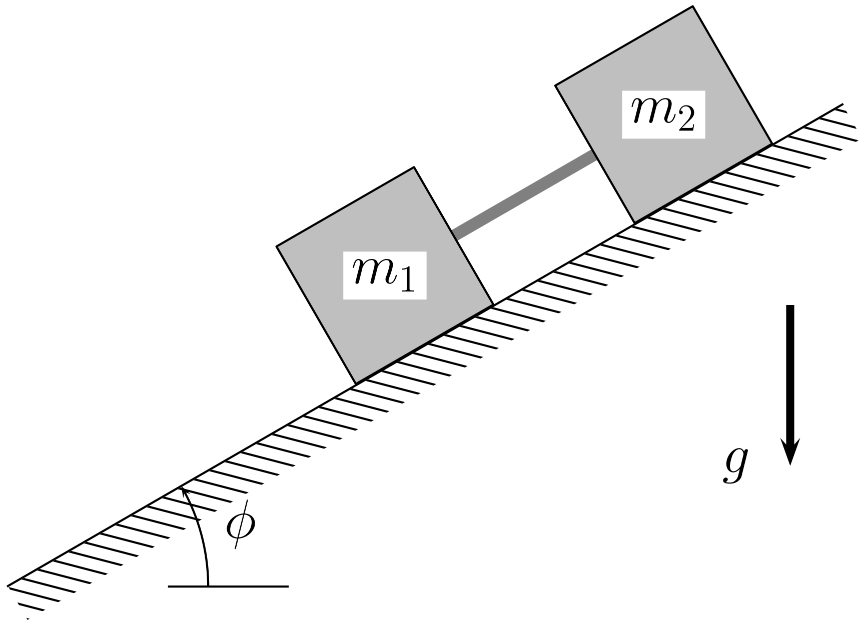
Engineering Mechanics: Dynamics

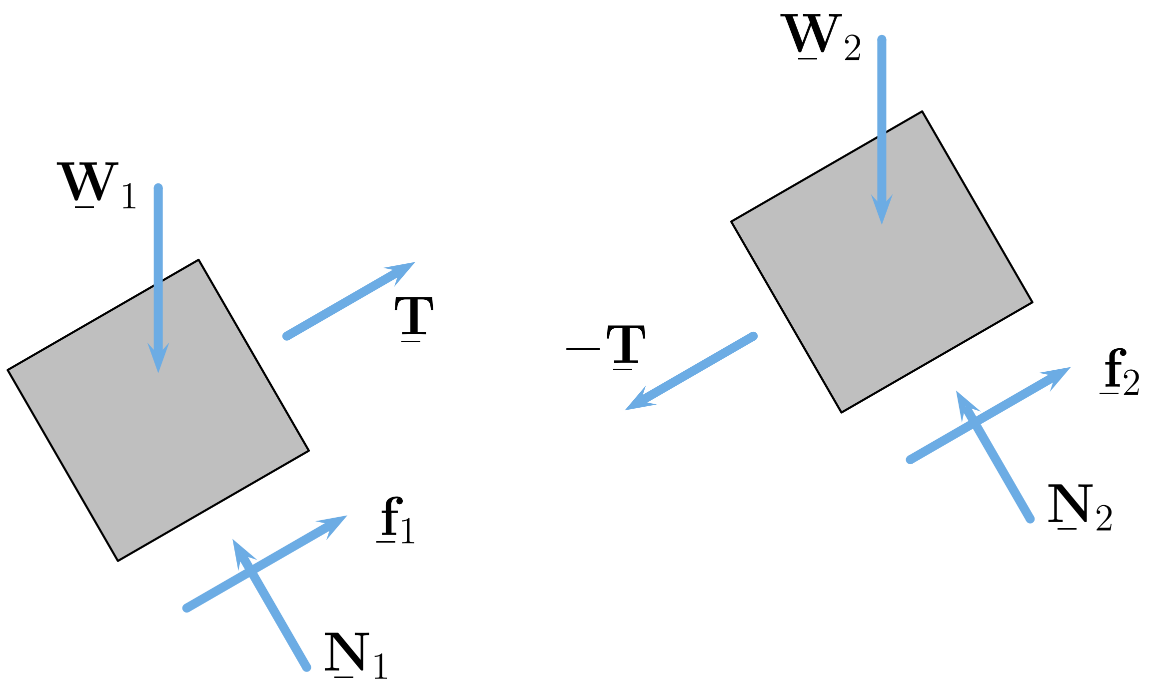
D. Dane Quinn, PhD

Department of Mechanical Engineering
The University of Akron
Akron OH 44325–3903 USA

quinn@uakron.edu
www.uakron.edu

Two blocks of mass m_1 and m_2 are connected by a massless rod and slide down a rough surface inclined at an angle ϕ. The coefficient of friction between each block and the surface is μ_1 and μ_2. Find the acceleration of the pair and the tension in the rod. What is the minimum angle $\phi_\star$ so that the blocks slide?





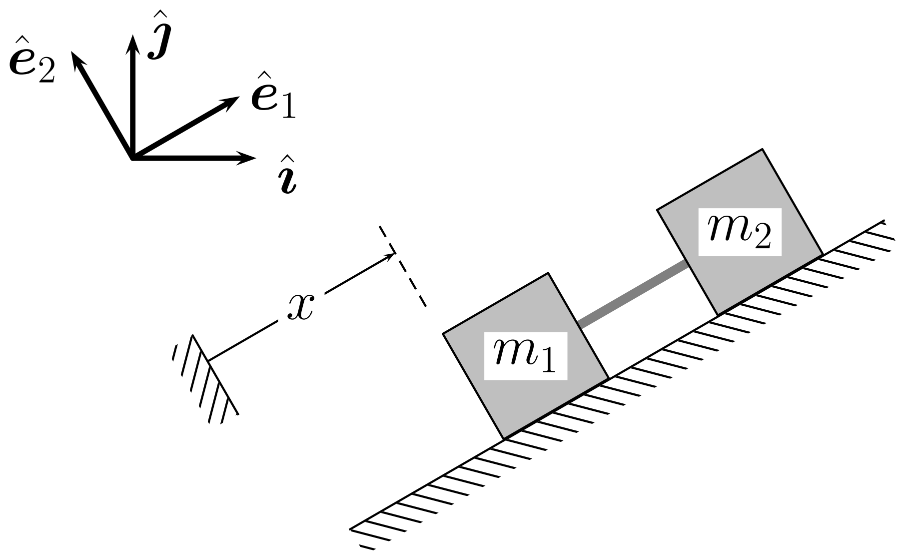
Forces acting on the blocks arise from

- Gravity ($\underline{\mathbf{W}}$)
- Friction ($\underline{\mathbf{f}}$)
- Normal to the surface ($\underline{\mathbf{N}}$)
- Tension in the rod ($\underline{\mathbf{T}}$)—internal so that the two components must balance

quinn@uakron.edu
www.uakron.edu

Coordinates and Directions

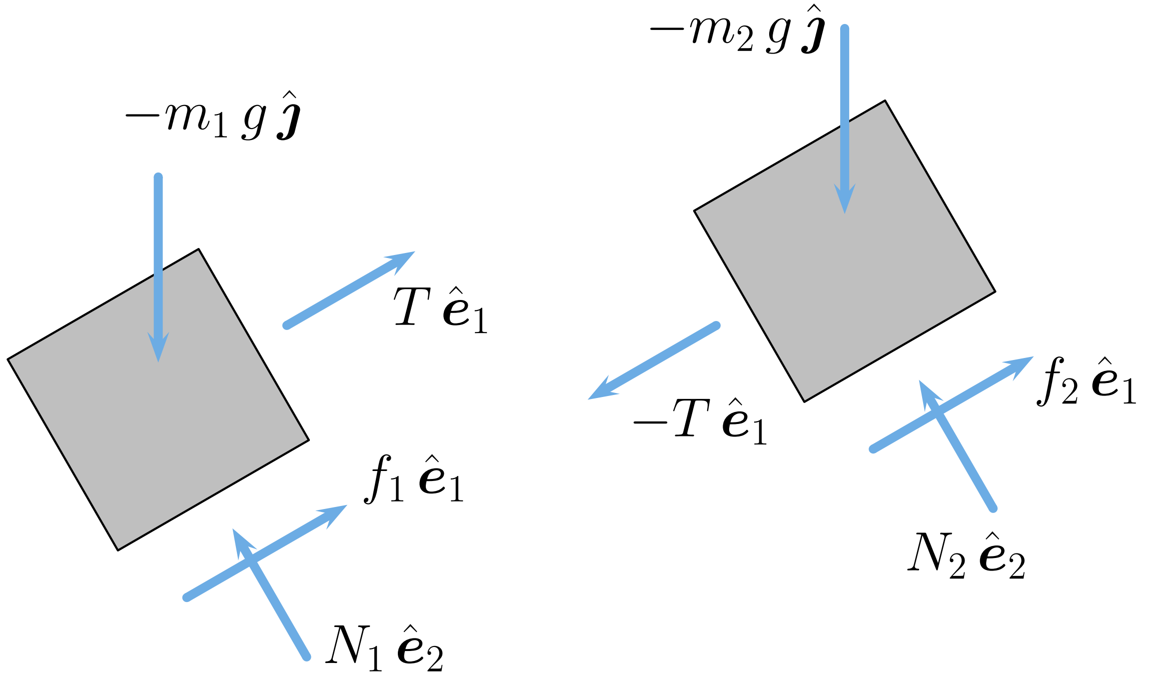
The motion of the system is only in the $\hat{e}_1$ direction, and the displacement of each block is identical, measured as x.



The directions $(\hat{\imath}, \hat{\jmath})$ and $(\hat{e}_1, \hat{e}_2)$ are related as

$$\hat{\imath} = \mathrm{C}_\phi \; \hat{e}_1 - \mathrm{S}_\phi \; \hat{e}_2, \qquad \hat{\jmath} = \mathrm{S}_\phi \; \hat{e}_1 + \mathrm{C}_\phi \; \hat{e}_2.$$

Kinematics/Free Body Diagram



With this, the acceleration of each block is

$$\underline{\mathbf{a}}_1 = \underline{\mathbf{a}}_2 = \ddot{x}\, \hat{e}_1,$$

while the free body diagram for each is shown to the left.

quinn@uakron.edu
www.uakron.edu

Equations of Motion

Applying linear momentum balance to each block

Block 1:

$$-m_1\, g\, \hat{\boldsymbol{\jmath}} + T\, \hat{\boldsymbol{e}}_1 + N_1\, \hat{\boldsymbol{e}}_2 + f_1\, \hat{\boldsymbol{e}}_1 = \sum \underline{\mathbf{F}}_1 = m_1\, \underline{\mathbf{a}}_1 = m_1\, \ddot{x}\, \hat{\boldsymbol{e}}_1,$$

Block 2:

$$-m_2\, g\, \hat{\boldsymbol{\jmath}} - T\, \hat{\boldsymbol{e}}_1 + N_2\, \hat{\boldsymbol{e}}_2 + f_2\, \hat{\boldsymbol{e}}_1 = \sum \underline{\mathbf{F}}_2 = m_2\, \underline{\mathbf{a}}_2 = m_2\, \ddot{x}\, \hat{\boldsymbol{e}}_1.$$

Therefore, taking components in the $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$ directions

Block 1:

$\hat{e}_1$ direction: $$-m_1\, g\, \mathrm{S}_\phi + T + f_1 = m_1\, \ddot{x},$$

$\hat{e}_2$ direction: $$-m_1\, g\, \mathrm{C}_\phi + N_1 = 0.$$

Block 2:

$\hat{e}_1$ direction: $$-m_2\, g\, \mathrm{S}_\phi - T + f_2 = m_2\, \ddot{x},$$

$\hat{e}_2$ direction: $$-m_2\, g\, \mathrm{C}_\phi + N_2 = 0.$$

quinn@uakron.edu
www.uakron.edu

From the above equations we can eliminate T and solve for the normal loads

$$(m_1 + m_2)\,\ddot{x} = -(m_1 + m_2)\,g\ \mathrm{S}_\phi + f_1 + f_2, \qquad N_1 = m_1\,g\ \mathrm{C}_\phi, \quad N_2 = m_2\,g\ \mathrm{C}_\phi\,.$$

so that

$$T = m_1\,\ddot{x} - f_1 + m_1\,g\ \mathrm{S}_\phi = \frac{m_1\,f_2 - m_2\,f_1}{m_1 + m_2}.$$

If the blocks are slipping down the plane, then $\dot{x} < 0$ and

$$\begin{aligned} f_1 &= \mu_1\,N_1 = \mu_1\,m_1\,g\ \mathrm{C}_\phi, \\ f_2 &= \mu_2\,N_2 = \mu_2\,m_2\,g\ \mathrm{C}_\phi, \end{aligned} \qquad \longrightarrow \qquad T = \frac{(\mu_2 - \mu_1)\,m_1\,m_2\,g\ \mathrm{C}_\phi}{m_1 + m_2},$$

so that

$$\ddot{x} = -g\ \mathrm{S}_\phi + \frac{(\mu_1\,m_1 + \mu_2\,m_2)\,g\ \mathrm{C}_\phi}{m_1 + m_2}, \qquad \longrightarrow \qquad \underline{\mathbf{a}}_1 = \underline{\mathbf{a}}_2 = \ddot{x}\,\hat{\boldsymbol{e}}_1,$$

If the blocks are instead sticking, then $\ddot{x} \equiv 0$, $|f_i| \le \mu_i\,N_i$. However, at the critical angle $\phi_\star$, $f_1 = \mu_1\,N_1$ and $f_2 = \mu_2\,N_2$, so that

$$0 = -(m_1 + m_2)\,g\ \mathrm{S}_{\phi_\star} + \mu_1\,m_1\,g\ \mathrm{C}_{\phi_\star} + \mu_2\,m_2\,g\ \mathrm{C}_{\phi_\star}, \qquad \longrightarrow \qquad \mathrm{T}_{\phi_\star} = \frac{\mu_1\,m_1 + \mu_2\,m_2}{m_1 + m_2}$$

quinn@uakron.edu
www.uakron.edu