Polar Motion

Engineering Mechanics: Dynamics

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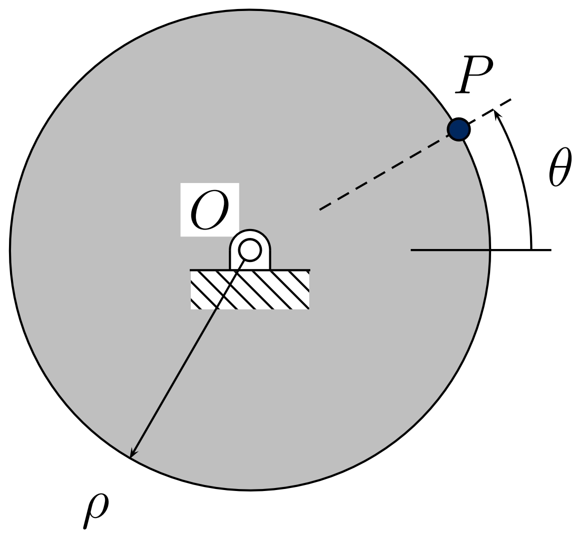
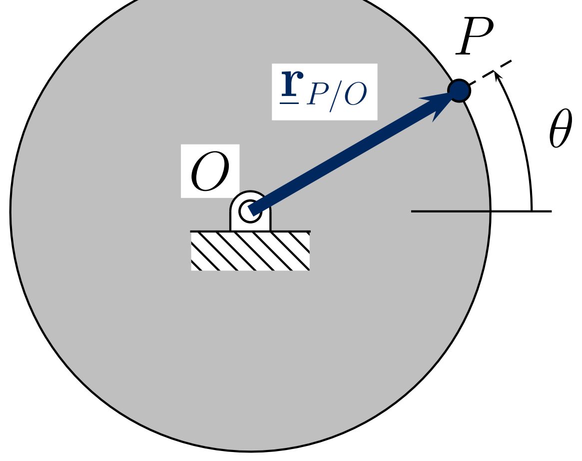
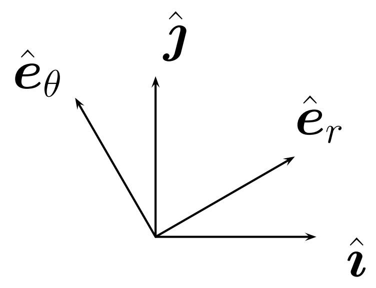
A disk of radius ρ is pinned at its center and rotates with constant angular speed $\dot{\theta} = \omega$. Find the velocity and acceleration of point P, and the speed of that point.







We define the directions $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta)$ and $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$, which are related as

$$\hat{\boldsymbol{e}}_r = \mathrm{C}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{S}_\theta\ \hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\imath}} = \mathrm{C}_\theta\ \hat{\boldsymbol{e}}_r - \mathrm{S}_\theta\ \hat{\boldsymbol{e}}_\theta,$$
$$\hat{\boldsymbol{e}}_\theta = -\mathrm{S}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{C}_\theta\ \hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\jmath}} = \mathrm{S}_\theta\ \hat{\boldsymbol{e}}_r + \mathrm{C}_\theta\ \hat{\boldsymbol{e}}_\theta.$$

The angle θ can be determined as

$$\int_0^t \left\{ \dot{\theta}(\tau) = \omega \right\} d\tau, \qquad (\theta(0) = 0)$$
$$\theta(t) = \omega\, t,$$

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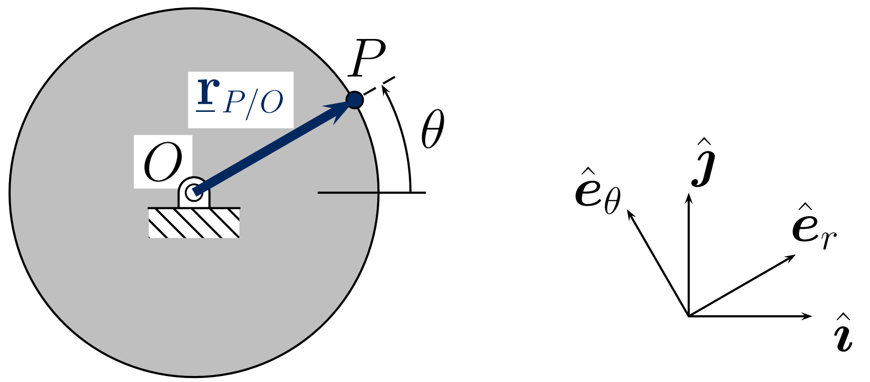
With the polar directions $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta)$, the position of P can be written as

$$\underline{\mathbf{r}}_{P/O} = r\,\hat{\boldsymbol{e}}_r \quad \longrightarrow \quad \begin{aligned} \underline{\mathbf{v}}_P &= \dot{r}\,\hat{\boldsymbol{e}}_r + r\,\dot{\theta}\,\hat{\boldsymbol{e}}_\theta, \\ \underline{\mathbf{a}}_P &= \left(\ddot{r} - r\,\dot{\theta}^2\right)\,\hat{\boldsymbol{e}}_r + \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\,\hat{\boldsymbol{e}}_\theta. \end{aligned}$$



For this system $r \equiv \rho$, so that

$$\dot{r} = 0, \qquad \ddot{r} = 0, \qquad \text{(Constant radius)}$$
$$\dot{\theta} = \omega, \qquad \ddot{\theta} = 0. \qquad \text{(Constant angular speed)}$$

So that

$$\left.\begin{aligned} \underline{\mathbf{v}}_P &= \rho\,\omega\,\hat{\boldsymbol{e}}_\theta, \\ \underline{\mathbf{a}}_P &= \left(-\rho\,\omega^2\right)\,\hat{\boldsymbol{e}}_r, \end{aligned}\right\} \longrightarrow \quad \text{Speed} \quad v \equiv \|\underline{\mathbf{v}}_P\| = \rho\,\omega.$$

Note that in terms of the directions $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$, the velocity and acceleration are

$$\begin{aligned} \underline{\mathbf{v}}_P &= \rho\,\omega\,\left(-\mathrm{S}_{\omega\,t}\;\hat{\boldsymbol{\imath}} + \mathrm{C}_{\omega\,t}\;\hat{\boldsymbol{\jmath}}\right) = \left(-\rho\,\omega\;\mathrm{S}_{\omega\,t}\right)\,\hat{\boldsymbol{\imath}} + \left(\rho\,\omega\;\mathrm{C}_{\omega\,t}\right)\,\hat{\boldsymbol{\jmath}}, \\ \underline{\mathbf{a}}_P &= -\rho\,\omega^2\,\left(\mathrm{C}_{\omega\,t}\;\hat{\boldsymbol{\imath}} + \mathrm{S}_{\omega\,t}\;\hat{\boldsymbol{\jmath}}\right) = \left(-\rho\,\omega^2\;\mathrm{C}_{\omega\,t}\right)\,\hat{\boldsymbol{\imath}} + \left(-\rho\,\omega^2\;\mathrm{S}_{\omega\,t}\right)\,\hat{\boldsymbol{\jmath}}. \end{aligned}$$

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