Polar Motion

Engineering Mechanics: Dynamics

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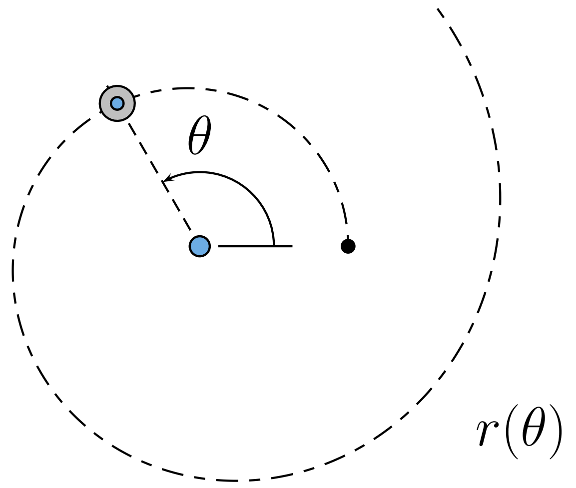
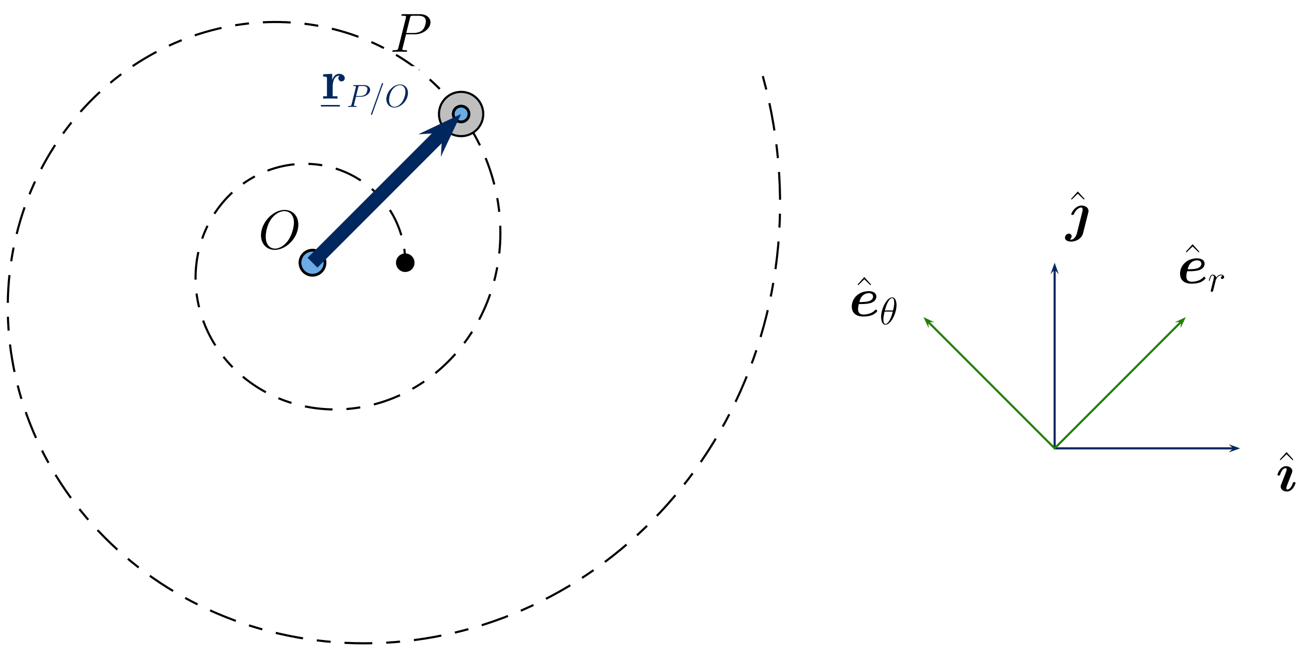
A particle moves on a helical path defined by

$$r(\theta) = k_0 + k_2\,\theta^2,$$

with $\dot{\theta} = \omega$. Find the position and velocity of the particle as a function of time if $\theta(0) = 0$.





Define the directions $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta)$, so that the position of P can be written as

$$\underline{\mathbf{r}}_{P/O} = r\,\hat{\boldsymbol{e}}_r, \qquad \underline{\mathbf{v}}_P = \dot{r}\,\hat{\boldsymbol{e}}_r + r\,\dot{\theta}\,\hat{\boldsymbol{e}}_\theta.$$

These directions can be related as

$$\hat{\boldsymbol{e}}_r = \mathrm{C}_\theta\;\hat{\boldsymbol{\imath}} + \mathrm{S}_\theta\;\hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\imath}} = \mathrm{C}_\theta\;\hat{\boldsymbol{e}}_r - \mathrm{S}_\theta\;\hat{\boldsymbol{e}}_\theta,$$
$$\hat{\boldsymbol{e}}_\theta = \mathrm{S}_\theta\;\hat{\boldsymbol{\imath}} - \mathrm{C}_\theta\;\hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\jmath}} = \mathrm{S}_\theta\;\hat{\boldsymbol{e}}_r + \mathrm{C}_\theta\;\hat{\boldsymbol{e}}_\theta.$$

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As a function of time

$$\int_0^t \left\{ \dot{\theta}(\tau) = \omega \right\} \, d\tau, \qquad \longrightarrow \qquad \theta(t) = \omega\, t.$$

and

$$\begin{aligned} r(t) &= k_0 + k_2\,\left(\theta(t)\right)^2, \qquad \longrightarrow \qquad \dot{r}(t) = \frac{d}{dt}\Big(r(\theta(t)) \Big) = \overbrace{\frac{dr}{d\theta}\,\frac{d\theta}{dt}}^{\text{chain rule}} = 2\,k_2\,\theta(t)\,\dot{\theta}(t), \\ &= k_0 + k_2\,(\omega\,t)^2, \qquad\qquad\qquad\qquad\qquad\qquad\qquad = 2\,k_2\,\omega^2\,t. \end{aligned}$$

Therefore

$$\begin{aligned} \underline{\mathbf{r}}_{P/O}(t) = r(t)\,\hat{\boldsymbol{e}}_r(t) &= \left(k_0 + k_2\,\omega^2\,t^2\right)\,\hat{\boldsymbol{e}}_r(t), \\ &= \left(\left(k_0 + k_2\,\omega^2\,t^2\right)\,\mathrm{C}_{\omega\,t}\right)\,\hat{\boldsymbol{\imath}} + \left(\left(k_0 + k_2\,\omega^2\,t^2\right)\,\mathrm{S}_{\omega\,t}\right)\,\hat{\boldsymbol{\jmath}}, \\ \underline{\mathbf{v}}_P(t) = \dot{r}(t)\,\hat{\boldsymbol{e}}_r(t) + r(t)\,\dot{\theta}(t)\,\hat{\boldsymbol{e}}_\theta(t) &= \left(2\,k_2\,\omega^2\,t\right)\,\hat{\boldsymbol{e}}_r(t) + \left(\left(k_0 + k_2\,\omega^2\,t^2\right)\,\omega\right)\,\hat{\boldsymbol{e}}_\theta(t), \\ &= \left(\left(2\,k_2\,\omega^2\,t\right)\,\mathrm{C}_{\omega\,t} - \left(\left(k_0 + k_2\,\omega^2\,t^2\right)\,\omega\right)\,\mathrm{S}_{\omega\,t}\right)\,\hat{\boldsymbol{\imath}} \\ &\quad + \left(\left(2\,k_2\,\omega^2\,t\right)\,\mathrm{S}_{\omega\,t} + \left(\left(k_0 + k_2\,\omega^2\,t^2\right)\,\omega\right)\,\mathrm{C}_{\omega\,t}\right)\,\hat{\boldsymbol{\jmath}}. \end{aligned}$$

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