Motion

Engineering Mechanics: Dynamics

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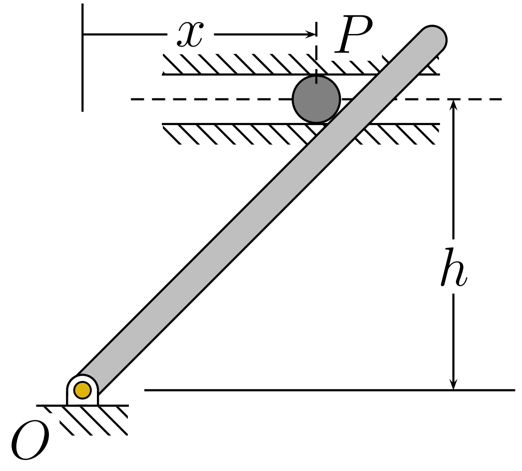
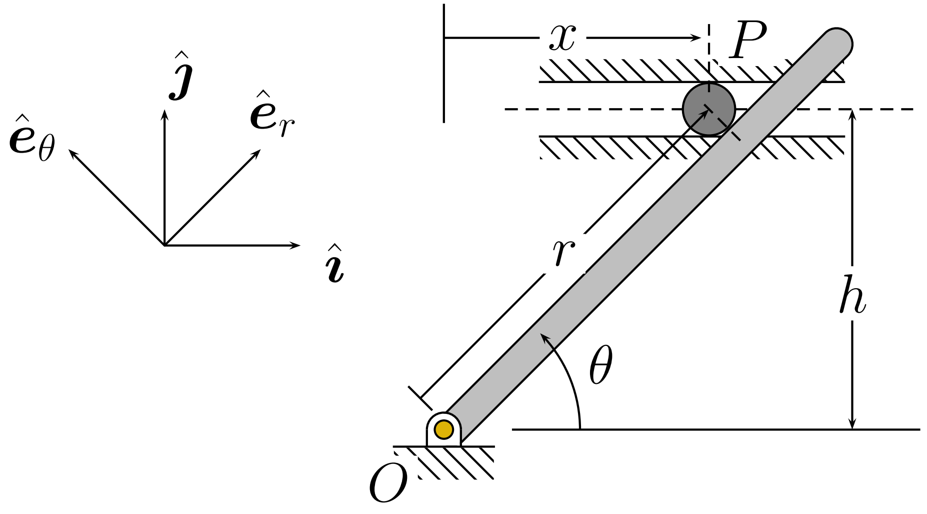
A ball moves along a horizontal slot, pushed by a rotating bar. Find the angular velocity and angular acceleration of the bar as a function of x so that the ball moves to the left with constant speed v.





The directions $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta)$ can be related to $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$ as

$$\hat{\boldsymbol{e}}_r = \mathrm{C}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{S}_\theta\ \hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\imath}} = \mathrm{C}_\theta\ \hat{\boldsymbol{e}}_r - \mathrm{S}_\theta\ \hat{\boldsymbol{e}}_\theta,$$
$$\hat{\boldsymbol{e}}_\theta = \mathrm{S}_\theta\ \hat{\boldsymbol{\imath}} - \mathrm{C}_\theta\ \hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\jmath}} = \mathrm{S}_\theta\ \hat{\boldsymbol{e}}_r + \mathrm{C}_\theta\ \hat{\boldsymbol{e}}_\theta,$$

and the angular velocity and acceleration of the bar are

$$\underline{\boldsymbol{\omega}}_{\mathcal{B}} = \dot{\theta}\,\hat{\boldsymbol{k}}, \qquad \underline{\boldsymbol{\alpha}}_{\mathcal{B}} = \ddot{\theta}\,\hat{\boldsymbol{k}}.$$

In terms of the Cartesian directions $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$

$$\underline{\mathbf{r}}_{P/O} = x\,\hat{\boldsymbol{\imath}} + h\,\hat{\boldsymbol{\jmath}}, \qquad \underline{\mathbf{v}}_P = \dot{x}\,\hat{\boldsymbol{\imath}} = -v\,\hat{\boldsymbol{\imath}}, \qquad \underline{\mathbf{a}}_P = \ddot{x}\,\hat{\boldsymbol{\imath}} = \underline{\mathbf{0}}.$$

With the polar directions $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta)$, the motion of the ball can be written as

$$\underline{\mathbf{r}}_{P/O} = r\,\hat{\boldsymbol{e}}_r, \qquad \underline{\mathbf{v}}_P = \dot{r}\,\hat{\boldsymbol{e}}_r + r\,\dot{\theta}\,\hat{\boldsymbol{e}}_\theta, \qquad \underline{\mathbf{a}}_P = \left(\ddot{r} - r\,\dot{\theta}^2\right)\,\hat{\boldsymbol{e}}_r + \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\,\hat{\boldsymbol{e}}_\theta.$$

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The velocity, $\underline{\mathbf{v}}_P$ can be written in both Cartesian and polar coordinates, so that

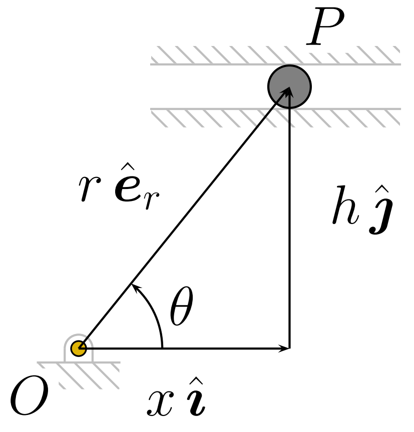
$$\dot{r}\,\hat{\boldsymbol{e}}_r + r\,\dot{\theta}\,\hat{\boldsymbol{e}}_\theta = \underline{\mathbf{v}}_P = \dot{x}\,\hat{\boldsymbol{\imath}},$$

$$= -v\,\hat{\boldsymbol{\imath}} = (-v\ \mathrm{C}_\theta)\ \hat{\boldsymbol{e}}_r + (v\ \mathrm{S}_\theta)\ \hat{\boldsymbol{e}}_\theta, \quad \longrightarrow \quad \dot{r} = -v\ \mathrm{C}_\theta, \quad \dot{\theta} = \frac{v\ \mathrm{S}_\theta}{r}.$$

Equating these descriptions of the position

$$x\,\hat{\boldsymbol{\imath}} + h\,\hat{\boldsymbol{\jmath}} = \underline{\mathbf{r}}_{P/O} = r\,\hat{\boldsymbol{e}}_r,$$

$$= (r\ \mathrm{C}_\theta)\ \hat{\boldsymbol{\imath}} + (r\ \mathrm{S}_\theta)\ \hat{\boldsymbol{\jmath}}, \quad \longrightarrow \quad r = \sqrt{x^2 + h^2}, \quad \mathrm{T}_\theta = \frac{h}{x},$$



so that

$$\mathrm{S}_\theta = \frac{h}{r} = \frac{h}{\sqrt{x^2 + h^2}}, \qquad \mathrm{C}_\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + h^2}}.$$

Therefore, in terms of x

$$\dot{r} = -\frac{v\,x}{r} = -\frac{v\,x}{\sqrt{x^2 + h^2}}, \quad \dot{\theta} = \frac{v\,h}{r^2} = \frac{v\,h}{x^2 + h^2}, \quad \longrightarrow \quad \underline{\boldsymbol{\omega}}_{\mathcal{B}} = \dot{\theta}\,\hat{\boldsymbol{k}} = \left(\frac{v\,h}{x^2 + h^2}\right)\,\hat{\boldsymbol{k}}$$

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The acceleration, $\underline{\mathbf{a}}_P$ can also be written in both Cartesian and polar coordinates, so that

$$\left(\ddot{r} - r\,\dot{\theta}^2\right)\,\hat{\boldsymbol{e}}_r + \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\,\hat{\boldsymbol{e}}_\theta = \underline{\mathbf{a}}_P = \ddot{x}\,\hat{\boldsymbol{\imath}},$$

$$= \underline{\mathbf{0}}, \quad \longrightarrow \quad \ddot{r} - r\,\dot{\theta}^2 = 0, \quad r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} = 0.$$

Solving for $\ddot{r}$ and $\ddot{\theta}$

$$\ddot{r} = r\,\dot{\theta}^2, \qquad \ddot{\theta} = \frac{2\,\dot{r}\,\dot{\theta}}{r},$$

$$= r\left(\frac{v\,h}{r^2}\right)^2 = \frac{v^2\,h^2}{r^3}, \qquad = \frac{2\,\left(-\frac{v\,x}{r}\right)\,\left(\frac{v\,h}{r^2}\right)}{r} = -\frac{2\,v^2\,x\,h}{r^4}.$$

Therefore, in terms of x

$$\ddot{r} = \frac{v^2\,h^2}{\left(x^2+h^2\right)^{3/2}}, \quad \ddot{\theta} = -\frac{2\,v^2\,h\,x}{\left(x^2+h^2\right)^2}, \quad \longrightarrow \quad \underline{\boldsymbol{\alpha}}_{\mathcal{B}} = \ddot{\theta}\,\hat{\boldsymbol{k}} = \left(-\frac{2\,v^2\,h\,x}{\left(x^2+h^2\right)^2}\right)\hat{\boldsymbol{k}}$$

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