Planar Dynamics

Engineering Mechanics: Dynamics

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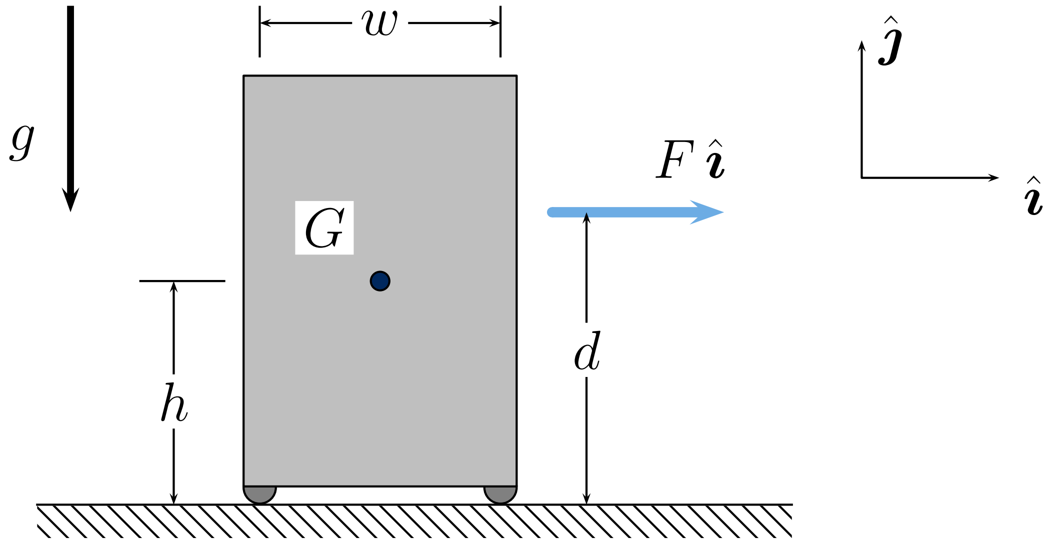
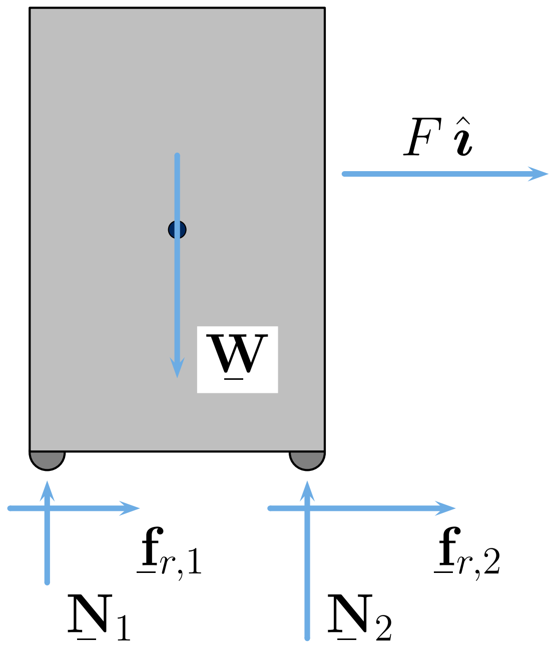
A block of mass m is in contact with a rough surface (coefficient of friction μ), and is pulled by a horizontal force of magnitude $F > 0$ applied at height d. The mass center of the block is located at height h and is centered between the supports, located a distance w apart. Find the maximum value of F so that the block does not tip when released from rest.





The block is subject to forces arising from

- Normal loads ($\underline{\mathbf{N}}_1$, $\underline{\mathbf{N}}_2$)) acting on each of the supports
- Friction ($\underline{\mathbf{f}}_{r,1}$, $\underline{\mathbf{f}}_{r,2}$)
- Weight ($\underline{\mathbf{W}}$)
- Applied force ($F\,\hat{\boldsymbol{\imath}}$)

Block rests on the surface $\longrightarrow$ normal forces must be in the $+\hat{\boldsymbol{\jmath}}$ direction

Assume that the block does not rotate and find the external load F_{max} so that either $\underline{\mathbf{N}}_1$ or $\underline{\mathbf{N}}_2$ becomes adhesive (in the $-\hat{\boldsymbol{\jmath}}$ direction).

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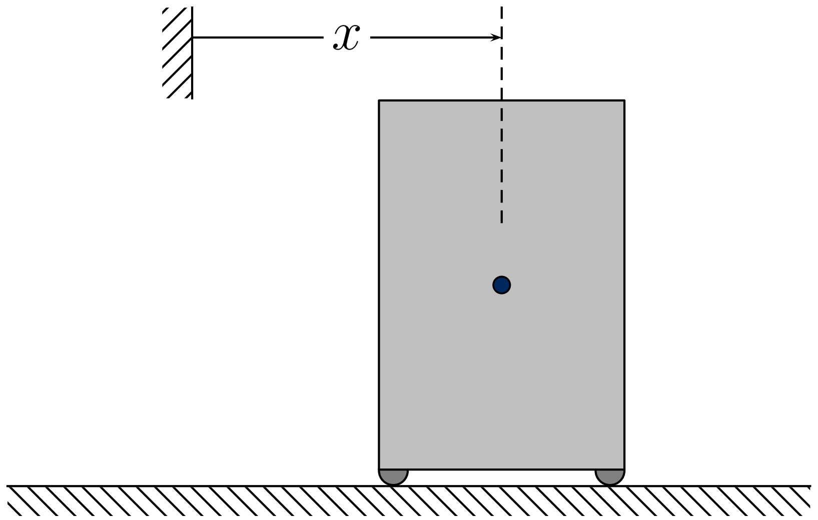
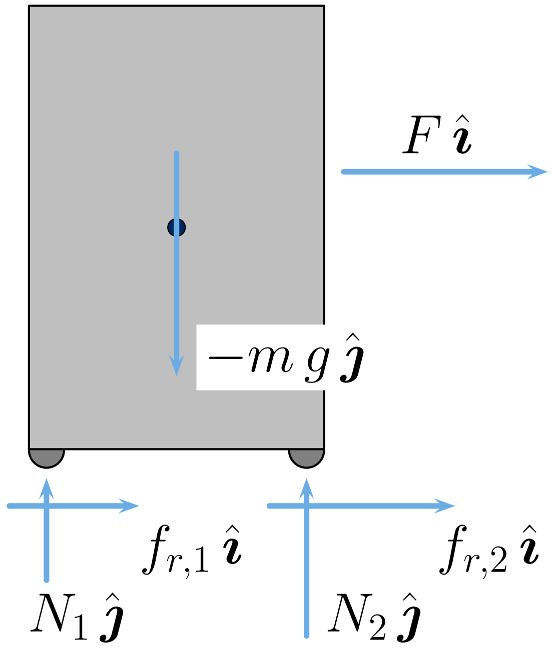
Coordinates and Directions/Kinematics



With the directions $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$, we measure the displacement of the block's center of mass as x. Since the block is assumed to not rotate, we need not measure its rotation. The acceleration of the block is then

$$\underline{\mathbf{a}}_G = \ddot{x}\,\hat{\boldsymbol{\imath}}.$$

Free Body Diagram



The forces can be identified in the free body diagram. In particular the friction force at each support can be written as

$$\begin{aligned} f_r &= -\mu\,N\,\operatorname{sgn}(\dot{x}), && |\dot{x}| > 0, \\ &= f_{\text{stick}}, && \dot{x} = 0, \end{aligned}$$

where f_{stick} is the force required to maintain equilibrium, and $|f_r| \leq \mu\,N$.

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Equations of Motion

Therefore, linear and angular momentum balance become

$\sum \underline{\mathbf{F}} = m\,\underline{\mathbf{a}}_G$

$$\big(F + f_{r,1} + f_{r,2}\big)\,\hat{\boldsymbol{\imath}} + (N_1 + N_2 - m\,g)\,\hat{\boldsymbol{\jmath}} = m\,\ddot{x}\,\hat{\boldsymbol{\imath}},$$

$\sum \underline{\mathbf{M}}_G = I_G\,\underline{\boldsymbol{\alpha}}_{\mathcal{B}}$

$$\left(-F\,(d-h) + f_{r,1}\,h + f_{r,2}\,h - N_1\,\frac{w}{2} + N_2\,\frac{w}{2}\right)\hat{\boldsymbol{k}} = \underline{\mathbf{0}}.$$

Taking components of these equations yields

$$F + f_{r,1} + f_{r,2} = m\,\ddot{x}, \qquad N_1 + N_2 - m\,g = 0,$$
$$-F\,(d-h) + f_{r,1}\,h + f_{r,2}\,h - N_1\,\frac{w}{2} + N_2\,\frac{w}{2} = 0,$$

in addition to the above description of the friction forces.

We must consider two cases

- Stick—the block does not move, so that $\ddot{x} \equiv 0$ and $|f_{r,i}| \leq \mu\,N_i$.
- Slip—the block moves, so that $|\dot{x}| > 0$ and $|f_{r,i}| = \mu\,N_i$

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Stick

If the block sticks, so that $\dot{x} = 0$, then $\ddot{x} = 0$ and the equations of motion reduce to

$$F + f_{r,1} + f_{r,2} = 0, \qquad N_1 + N_2 - m\,g = 0,$$
$$-F\,(d - h) + f_{r,1}\,h + f_{r,2}\,h - N_1\,\frac{w}{2} + N_2\,\frac{w}{2} = 0.$$

Note there are four unknowns $(f_{r,1}, f_{r,2}, N_1, N_2)$ and only three equations. However, the frictional terms always appear together, so that we solve for $f_{r,1} + f_{r,2}$ as

$$f_{r,1} + f_{r,2} = -F,$$

and the remaining equations become

$$N_1 + N_2 = m\,g, \qquad -F\,(d - h) - F\,h - (N_1 - N_2)\,\frac{w}{2} = 0.$$

Thus, solving for N_1 and N_2 yields

$$N_1 = \frac{m\,g}{2} - \frac{F\,d}{w}, \qquad N_2 = \frac{m\,g}{2} + \frac{F\,d}{w},$$

while

$$F = |f_{r,1} + f_{r,2}| \leq \mu\,(N_1 + N_2) = \mu\,m\,g$$

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Both N_1 and N_2 must be positive. With $F > 0$, then $N_2 > 0$ but for $F = F_\star$, then $N_1 = 0$, so that the block begins to tip. Therefore

$$N_1 = \frac{m\,g}{2} - \frac{F_\star\,d}{w} = 0, \qquad \longrightarrow \qquad F_\star = \frac{m\,g\,w}{2\,d}.$$

Slip

If $F > \mu\,m\,g$ then the block moves in the direction of the force, so that $\dot{x} > 0$ and $\ddot{x} > 0$. With this, $f_{r,1} = -\mu\,N_1$ and $f_{r,2} = -\mu\,N_2$ and the equations of motion can be written as

$$F - \mu\,(N_1 + N_2) = m\,\ddot{x}, \qquad N_1 + N_2 = m\,g,$$
$$-F\,(d-h) - \mu\,(N_1 + N_2)\,h - (N_1 - N_2)\,\frac{w}{2} = 0.$$

As a result, we solve for $\ddot{x}$, N_1, and N_2 as

$$\ddot{x} = F - \mu\,m\,g,$$
$$N_1 = \frac{m\,g}{2} - \frac{F\,(d-h) + \mu\,m\,g\,h}{w}, \qquad N_2 = \frac{m\,g}{2} + \frac{F\,(d-h) + \mu\,m\,g\,h}{w}$$

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As before, both N_1 and N_2 must be positive for steady sliding. Depending on the sign of $d-h$, either of these normal forces could vanish.

$d-h>0$

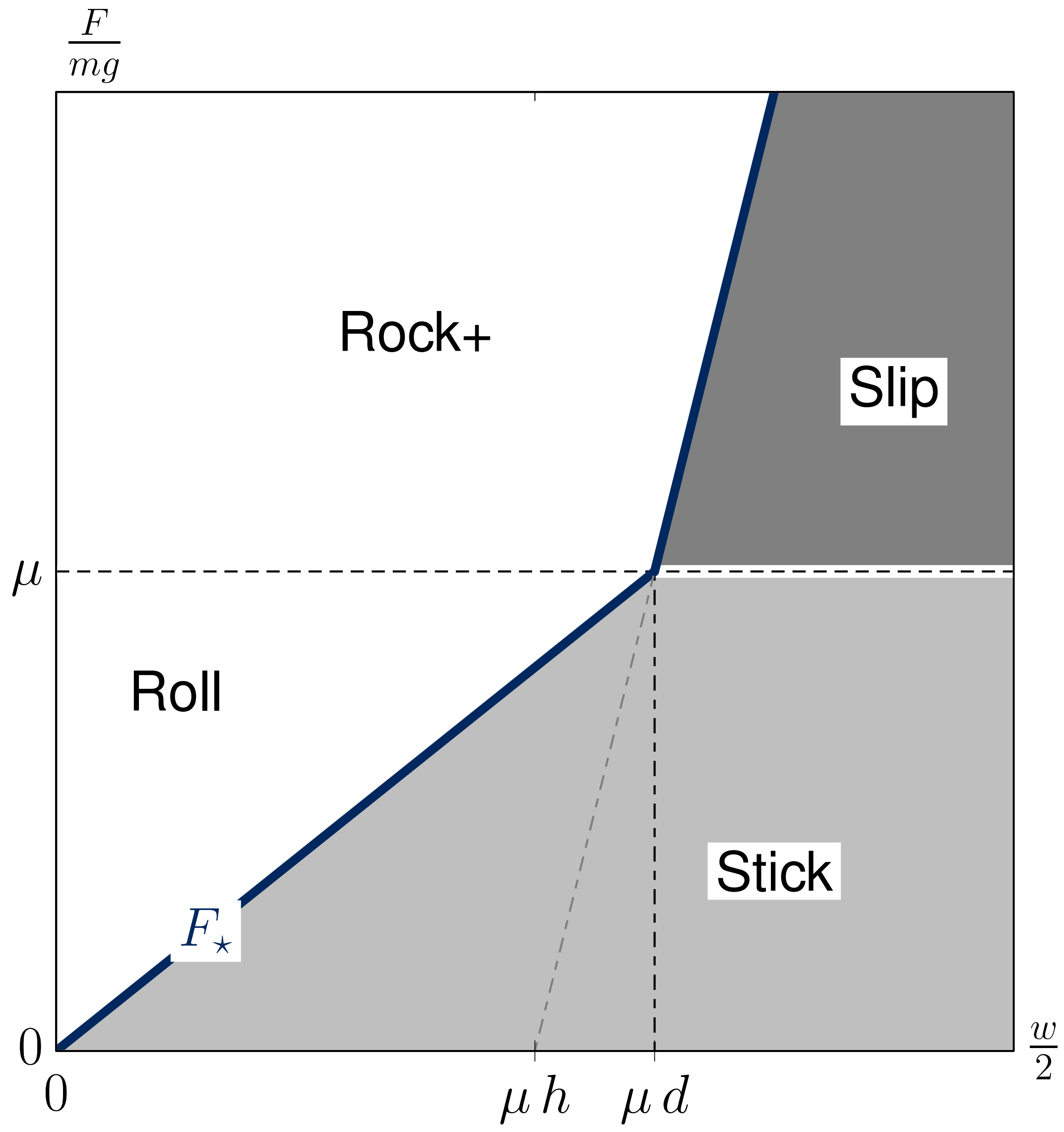
$F \le \mu\, m\, g$:

$$\frac{F_\star}{mg} = \frac{w/2}{d}.$$

$F > \mu\, m\, g$: N_2 is always positive, but $N_1 = 0$ for $F = F_\star$ with

$$N_1 = \frac{m\,g}{2} - \frac{F_\star\,(d-h) + \mu\, m\, g\, h}{w} = 0,$$

$$\longrightarrow \qquad F_\star = \frac{m\,g}{d-h}\left(\frac{w}{2} - \mu\, h\right),$$



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$d - h < 0$

For $F > 0$ and $d - h < 0$, then

$$N_1 = \frac{m\,g}{2} + \frac{F\,(h-d) - \mu\,m\,g\,h}{w}, \qquad N_2 = \frac{m\,g}{2} - \frac{F\,(h-d) - \mu\,m\,g\,h}{w}$$

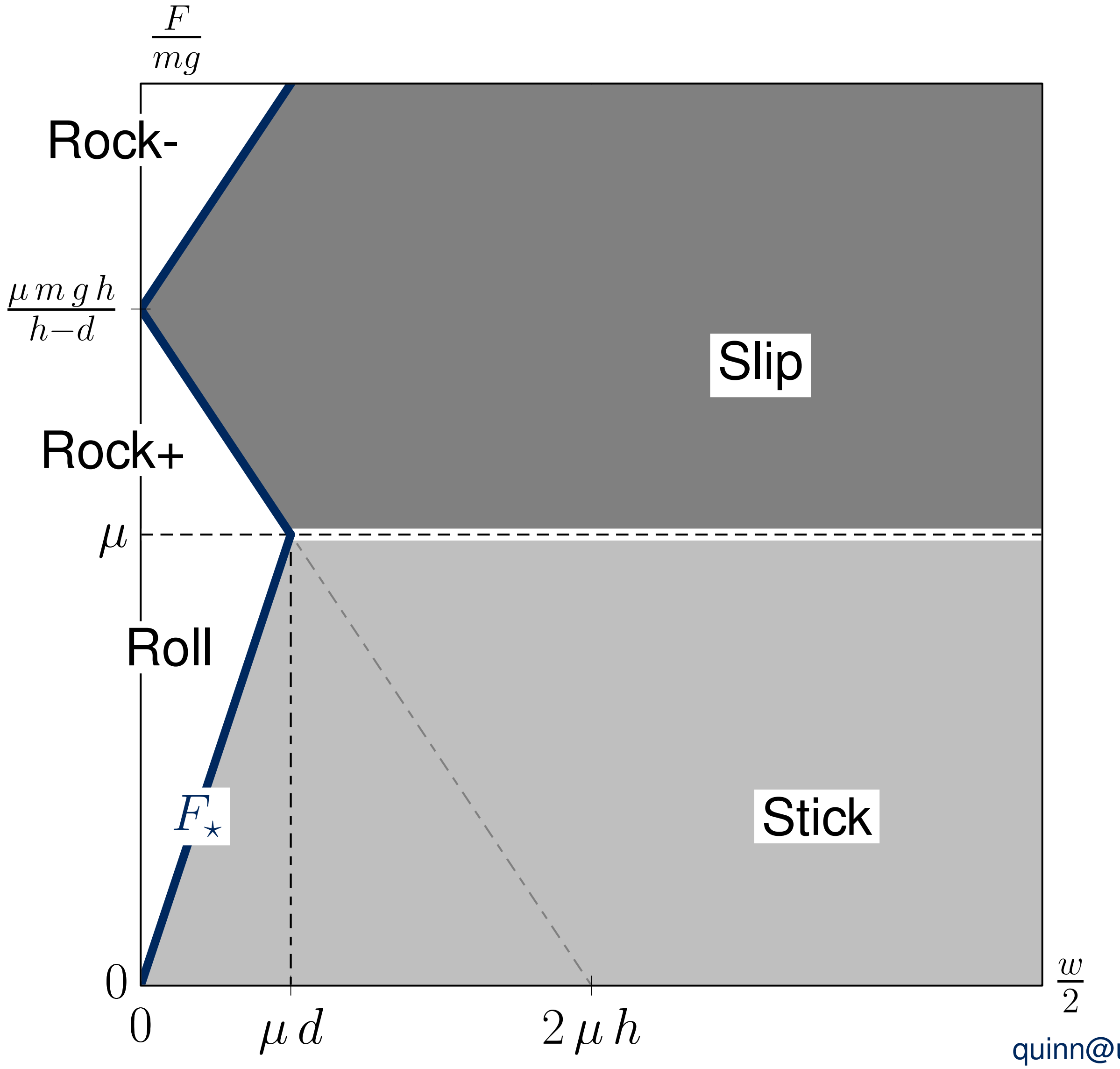
$F \le \mu\,m\,g$:

$$\frac{F_\star}{mg} = \frac{w/2}{d}.$$

$F > \mu\,m\,g$: Either $N_1 = 0$ or $N_2 = 0$ can vanish for some value of $F = F_\star$

$$N_1 = \frac{m\,g}{2} + \frac{F_\star\,(h-d) - \mu\,m\,g\,h}{w} = 0, \qquad \longrightarrow \qquad F_\star = \frac{m\,g}{h-d}\left(\mu\,h - \frac{w}{2}\right),$$

$$N_2 = \frac{m\,g}{2} - \frac{F_\star\,(h-d) - \mu\,m\,g\,h}{w} = 0, \qquad \longrightarrow \qquad F_\star = \frac{m\,g}{h-d}\left(\mu\,h + \frac{w}{2}\right),$$



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