Rigid Body Dynamics

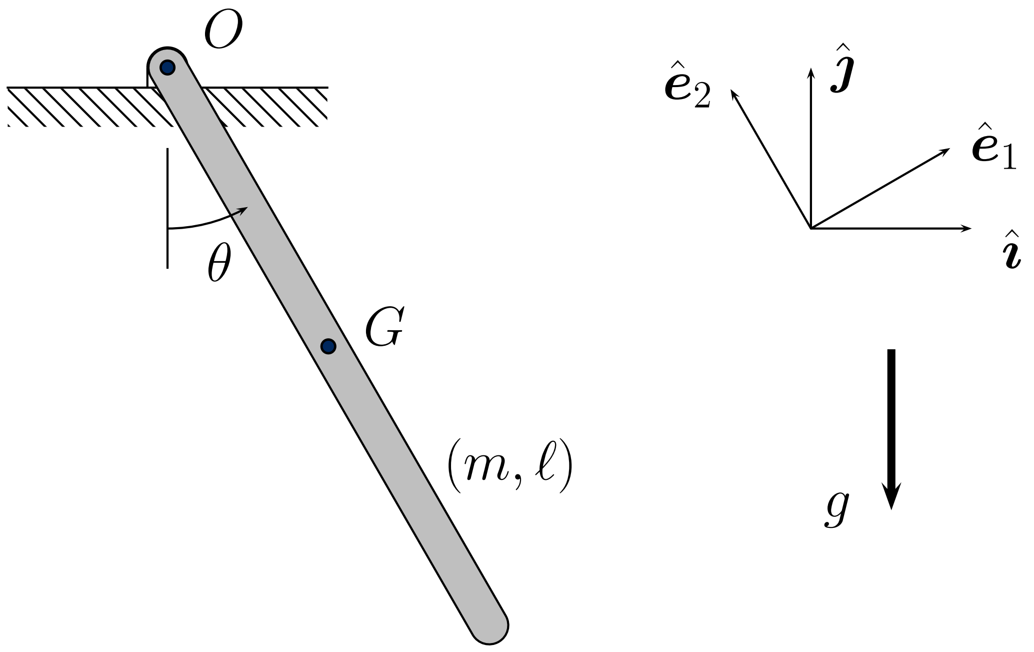
Engineering Mechanics: Dynamics

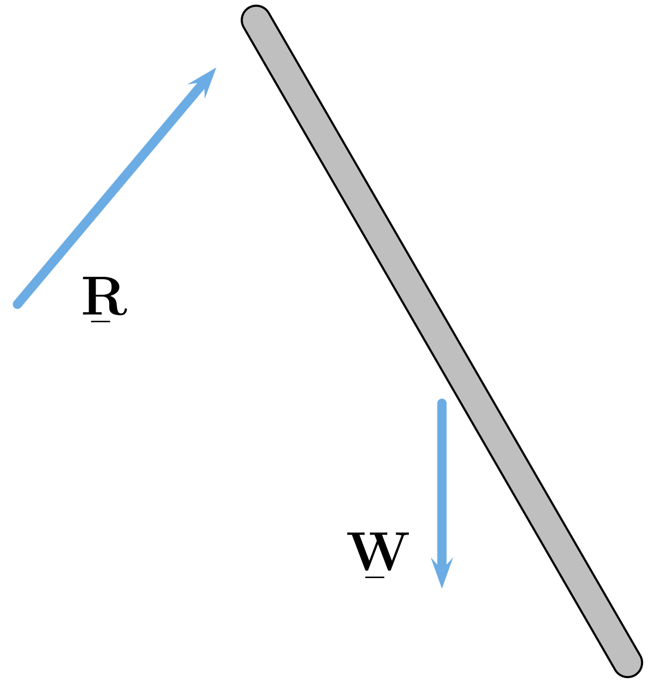
D. Dane Quinn, PhD

Department of Mechanical Engineering
The University of Akron
Akron OH 44325–3903 USA

quinn@uakron.edu
www.uakron.edu

The bar of mass m and length ℓ is pinned at O and is released from rest at an angle θ. Find its resulting angular acceleration at the instant it is released, as well as the reaction force at the pin.





The bar is subject to forces arising from the

- Reaction force ($\underline{\mathbf{R}}$) at the pin
- Weight ($\underline{\mathbf{W}}$)

Coordinates and Directions

The coordinate θ can be used to describe the configuration of the bar, so that

$$\hat{\boldsymbol{\imath}} = \mathrm{C}_\theta\ \hat{\boldsymbol{e}}_1 - \mathrm{S}_\theta\ \hat{\boldsymbol{e}}_2, \qquad \hat{\boldsymbol{\jmath}} = \mathrm{S}_\theta\ \hat{\boldsymbol{e}}_1 + \mathrm{C}_\theta\ \hat{\boldsymbol{e}}_2.$$

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Kinematics

The angular acceleration of the bar is $\underline{\boldsymbol{\alpha}}_{\mathcal{B}} = \ddot{\theta}\,\hat{\boldsymbol{k}}$, while the acceleration of the mass center G can be written as

$$\underline{\mathbf{a}}_G = \underline{\mathbf{a}}_O + \underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{G/O} + \underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \left(\underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{G/O}\right).$$

O is pinned in the ground, so that $\underline{\mathbf{a}}_O = \underline{\mathbf{0}}$, so that $\underline{\mathbf{a}}_G$ can be written as

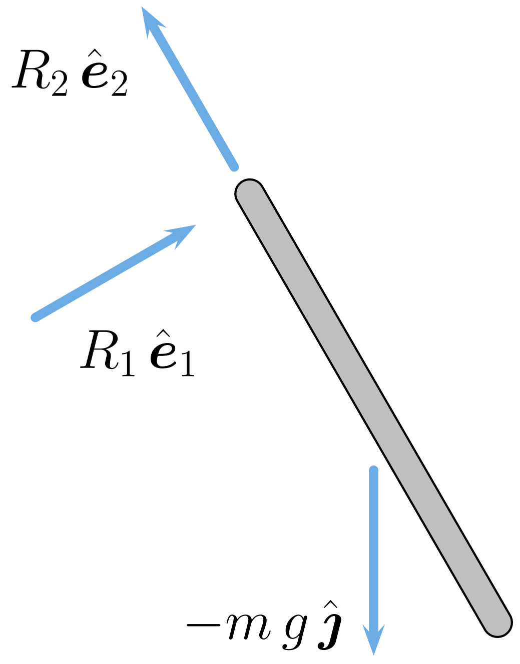
$$\underline{\mathbf{a}}_G = \left(\ddot{\theta}\,\hat{\boldsymbol{k}}\right) \times \left(-\frac{\ell}{2}\,\hat{\boldsymbol{e}}_2\right) + \left(\dot{\theta}\,\hat{\boldsymbol{k}}\right) \times \left(\left(\dot{\theta}\,\hat{\boldsymbol{k}}\right) \times \left(-\frac{\ell}{2}\,\hat{\boldsymbol{e}}_2\right)\right) = \frac{\ell\,\ddot{\theta}}{2}\,\hat{\boldsymbol{e}}_1 + \frac{\ell\,\dot{\theta}^2}{2}\,\hat{\boldsymbol{e}}_2.$$

Note that if the bar is released from rest $\underline{\boldsymbol{\omega}}_{\mathcal{B}} = \underline{\mathbf{0}}$.

Free Body Diagram

The free body diagram is shown to the right. Note that the unknown reaction force $\underline{\mathbf{R}}$ has been decomposed into components in the $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$ directions, so that

$$\underline{\mathbf{R}} = R_1\,\hat{\boldsymbol{e}}_1 + R_2\,\hat{\boldsymbol{e}}_2.$$



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Equations of Motion

Linear momentum balance for the bar can be written as

$$\sum \underline{\mathbf{F}} = R_1\,\hat{\boldsymbol{e}}_1 + R_2\,\hat{\boldsymbol{e}}_2 - m\,g\,\hat{\boldsymbol{\jmath}} = \frac{m\,\ell\,\ddot{\theta}}{2}\,\hat{\boldsymbol{e}}_1 + \frac{m\,\ell\,\dot{\theta}^2}{2}\,\hat{\boldsymbol{e}}_2 = m\,\underline{\mathbf{a}}_G,$$

and taking components

$\hat{\boldsymbol{e}}_1$ direction: $$R_1 - m\,g\,\mathrm{S}_\theta = \frac{m\,\ell\,\ddot{\theta}}{2},$$

$\hat{\boldsymbol{e}}_2$ direction: $$R_2 - m\,g\,\mathrm{C}_\theta = \frac{m\,\ell\,\dot{\theta}^2}{2},$$

Because the bar is pinned at O we take angular momentum balance about that point. The moment from the weight becomes

$$\underline{\mathbf{M}}_{\text{gravity}} = \underline{\mathbf{r}}_{G/O} \times \underline{\mathbf{W}} = \left(-\frac{\ell}{2}\,\hat{\boldsymbol{e}}_2\right) \times (-m\,g\,\hat{\boldsymbol{\jmath}}) = -\frac{m\,g\,\ell}{2}\,\mathrm{S}_\theta\,\hat{\boldsymbol{k}},$$

so that angular momentum balance becomes

$$\sum \underline{\mathbf{M}}_O = -\frac{m\,g\,\ell}{2}\,\mathrm{S}_\theta\,\hat{\boldsymbol{k}} = \frac{m\,\ell^2}{3}\,\ddot{\theta}\,\hat{\boldsymbol{k}} = I_O\,\underline{\boldsymbol{\alpha}}_{\mathcal{B}}$$

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Angular momentum balance yields

$$\ddot{\theta} = -\frac{3\,g}{2\,\ell}\,\mathrm{S}_\theta, \quad \longrightarrow \quad \underline{\boldsymbol{\alpha}}_{\mathcal{B}} = \left(-\frac{3\,g}{2\,\ell}\,\mathrm{S}_\theta\right)\hat{\boldsymbol{k}},$$

and finally the components of the reaction force are

$$\begin{aligned} R_1 &= m\,g\,\mathrm{S}_\theta + \frac{m\,\ell}{2}\,\ddot{\theta}, \qquad R_2 = m\,g\,\mathrm{C}_\theta + \frac{m\,\ell}{2}\,\dot{\theta}^2. \\ &= \frac{m\,g}{4}\,\mathrm{S}_\theta, \end{aligned}$$

If the system is released from rest,then $\dot{\theta} = 0$, so that the reaction force is

$$\underline{\mathbf{R}} = R_1\,\hat{\boldsymbol{e}}_1 + R_2\,\hat{\boldsymbol{e}}_2 = \left(\frac{m\,g}{4}\,\mathrm{S}_\theta\right)\hat{\boldsymbol{e}}_1 + (m\,g\,\mathrm{C}_\theta)\,\hat{\boldsymbol{e}}_2.$$

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