Planar Dynamics

Engineering Mechanics: Dynamics

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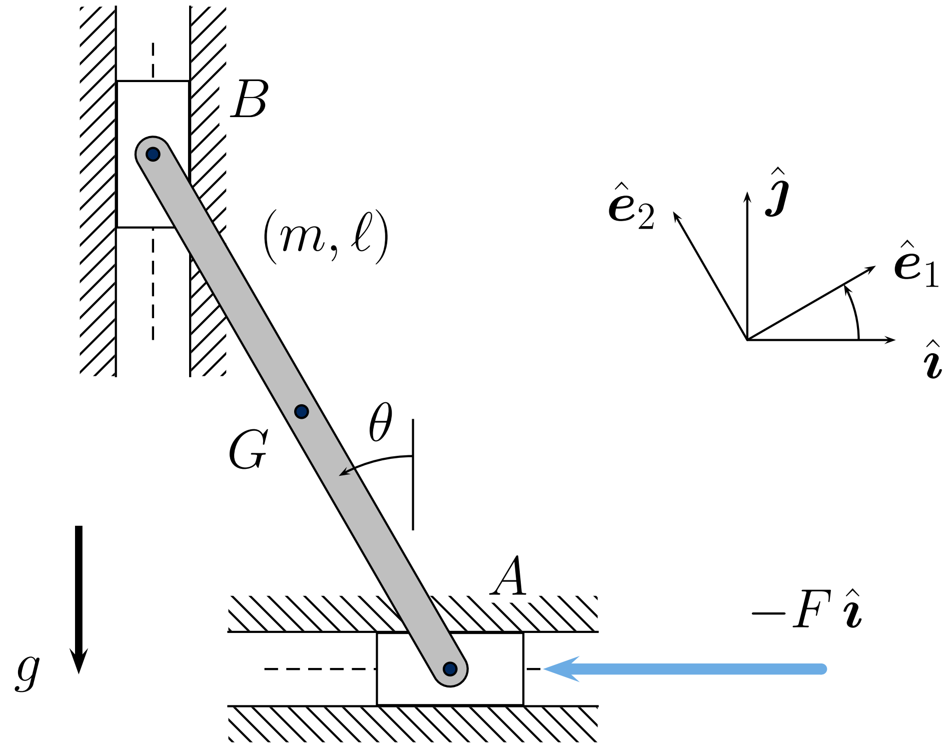
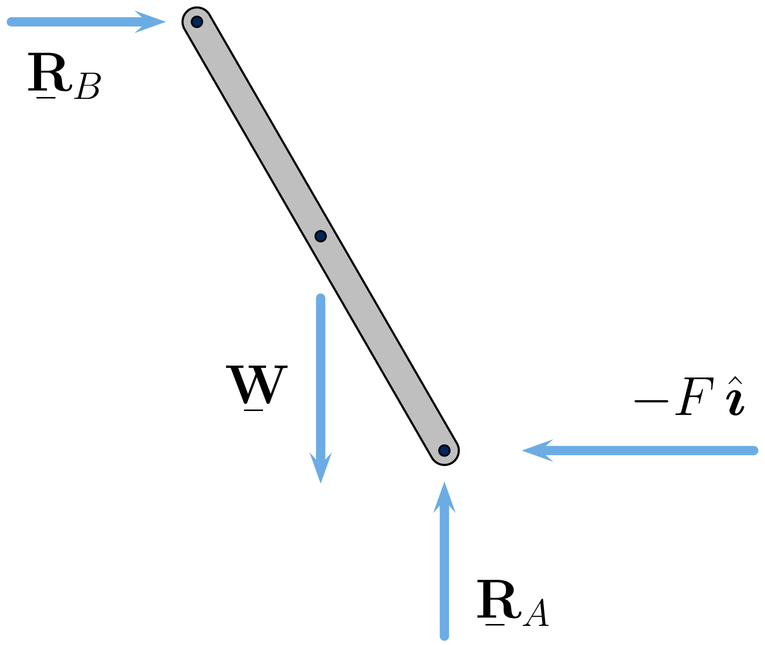
The bar of mass m and length ℓ slides along the smooth horizontal and vertical slots and the (massless) block at A is subject to a force $-F\,\hat{\boldsymbol{\imath}}$. Find the resulting angular acceleration of the bar if it is released from rest at angle θ.





The forces acting on the bar arise from

- Reaction forces ($\underline{\mathbf{R}}$) on the blocks
- Weight ($\underline{\mathbf{W}}$) of the bar
- External force ($-F\,\hat{\boldsymbol{\imath}}$)

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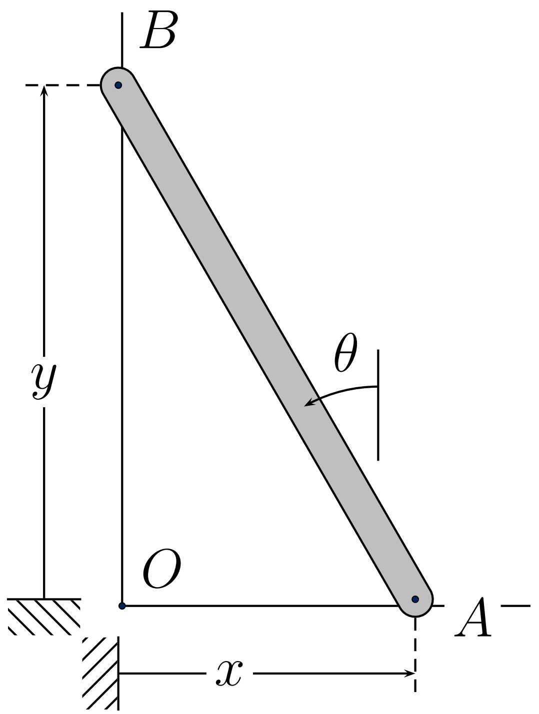
Coordinates and Directions



Define the directions $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$ fixed in the bar and $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$ fixed in the ground, so that

$$\hat{\boldsymbol{e}}_1 = \mathrm{C}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{S}_\theta\ \hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{e}}_2 = -\,\mathrm{S}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{C}_\theta\ \hat{\boldsymbol{\jmath}},$$

where the rotation of the bar is defined by θ. Also, the blocks are constrained to move along the slots, so that

$$\underline{\mathbf{r}}_{A/O} = x\,\hat{\boldsymbol{\imath}}, \qquad \underline{\mathbf{r}}_{B/O} = y\,\hat{\boldsymbol{\jmath}}.$$

Kinematics

Because of the sliding constraints this is a single degree-of-freedom system, so that the motion of the mass center can be related to the rotation of the bar. Note that

$$\underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{B/A} + \underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \left(\underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{B/A}\right),$$

$$\ddot{y}\,\hat{\boldsymbol{\jmath}} = \ddot{x}\,\hat{\boldsymbol{\imath}} + \left(\ddot{\theta}\,\hat{\boldsymbol{k}}\right) \times (\ell\,\hat{\boldsymbol{e}}_2) + \underline{\mathbf{0}} = \ddot{x}\,\hat{\boldsymbol{\imath}} - \ell\,\ddot{\theta}\,\hat{\boldsymbol{e}}_1, \quad \longrightarrow$$

$\hat{\boldsymbol{\imath}}$ direction: $0 = \ddot{x} - \ell\,\ddot{\theta}\ \mathrm{C}_\theta,$

$\hat{\boldsymbol{\jmath}}$ direction: $\ddot{y} = -\ell\,\ddot{\theta}\ \mathrm{S}_\theta\,.$

Therefore

$$\ddot{x} = \ell\,\ddot{\theta}\ \mathrm{C}_\theta, \qquad \ddot{y} = -\ell\,\ddot{\theta}\ \mathrm{S}_\theta$$

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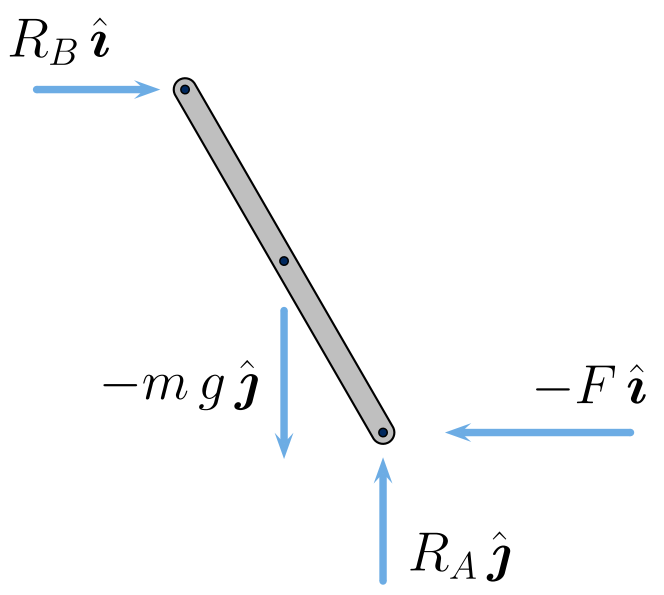
Finally,

$$\underline{\mathbf{a}}_G = \underline{\mathbf{a}}_A + \underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{G/A} + \underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \left(\underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{G/A}\right),$$
$$= \ddot{x}\,\hat{\boldsymbol{\imath}} + \left(\ddot{\theta}\,\hat{\boldsymbol{k}}\right) \times \left(\frac{\ell}{2}\,\hat{\boldsymbol{e}}_2\right) + \underline{\mathbf{0}},$$
$$= \left(\ell\,\ddot{\theta}\,\mathrm{C}_\theta\right)\hat{\boldsymbol{\imath}} - \frac{\ell}{2}\ddot{\theta}\,\hat{\boldsymbol{e}}_1 = \left(\frac{\ell}{2}\ddot{\theta}\,\mathrm{C}_\theta\right)\hat{\boldsymbol{\imath}} + \left(-\frac{\ell}{2}\ddot{\theta}\,\mathrm{S}_\theta\right)\hat{\boldsymbol{\jmath}}.$$

Free Body Diagram/Equations of Motion



Momentum balance for the bar can be written as

$$\sum \underline{\mathbf{F}} = (R_B - F)\,\hat{\boldsymbol{\imath}} + (R_A - m\,g)\,\hat{\boldsymbol{\jmath}}$$
$$= m\left(\left(\frac{\ell}{2}\ddot{\theta}\,\mathrm{C}_\theta\right)\hat{\boldsymbol{\imath}} + \left(-\frac{\ell}{2}\ddot{\theta}\,\mathrm{S}_\theta\right)\hat{\boldsymbol{\jmath}}\right) = m\,\underline{\mathbf{a}}_G,$$

$$\sum \underline{\mathbf{M}}_G = \left(\frac{\ell\,\mathrm{S}_\theta}{2}R_A - \frac{\ell\,\mathrm{C}_\theta}{2}(F + R_B)\right)\hat{\boldsymbol{k}} = \frac{m\,\ell^2}{12}\ddot{\theta}\,\hat{\boldsymbol{k}} = I_G\,\underline{\boldsymbol{\alpha}}_{\mathcal{B}}.$$

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Taking components of these in the $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}})$ directions

$$R_B - F = \frac{m\,\ell\,\mathrm{C}_\theta}{2}\,\ddot{\theta}, \qquad R_A - m\,g = -\frac{m\,\ell\,\mathrm{S}_\theta}{2}\,\ddot{\theta}, \qquad \frac{\ell\,\mathrm{S}_\theta}{2}\,R_A - \frac{\ell\,\mathrm{C}_\theta}{2}\,(F + R_B) = \frac{m\,\ell^2}{12}\,\ddot{\theta}.$$

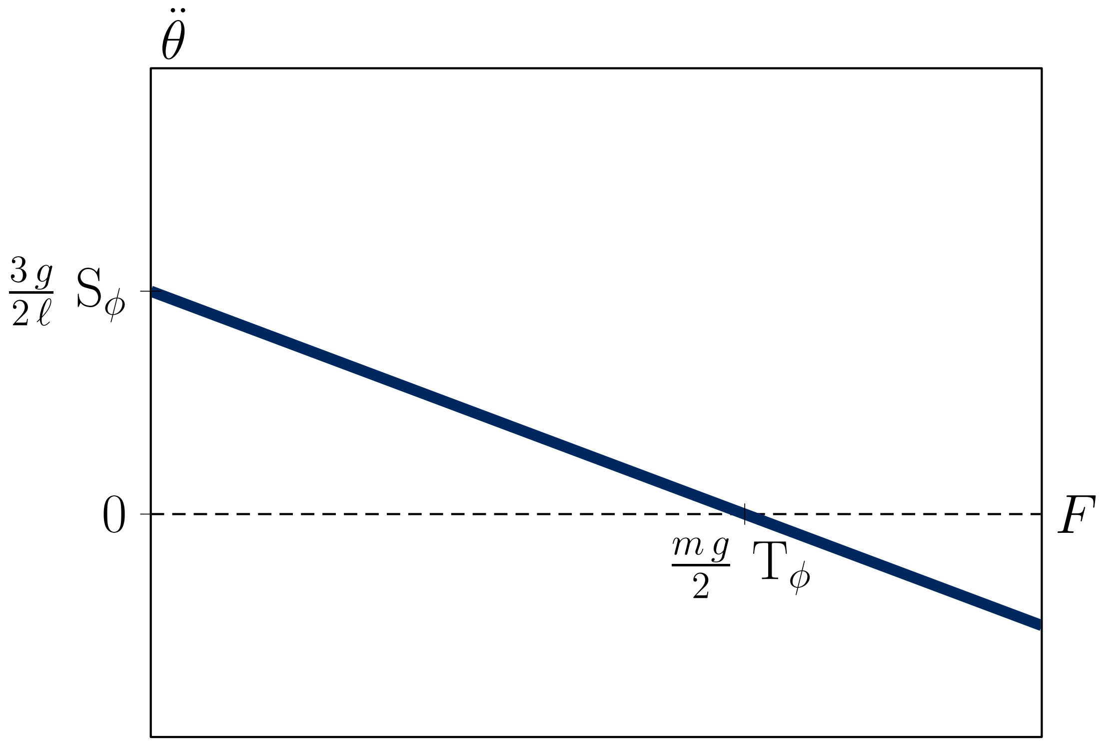
Therefore

$$R_A = m\,g - \frac{m\,\ell\,\mathrm{S}_\theta}{2}\,\ddot{\theta}, \quad R_B = F + \frac{m\,\ell\,\mathrm{C}_\theta}{2}\,\ddot{\theta},$$

$$\longrightarrow \quad \frac{\ell\,\mathrm{S}_\theta}{2}\left(m\,g - \frac{m\,\ell\,\mathrm{S}_\theta}{2}\,\ddot{\theta}\right) - \frac{\ell\,\mathrm{C}_\theta}{2}\left(F + F + \frac{m\,\ell\,\mathrm{C}_\theta}{2}\,\ddot{\theta}\right) = \frac{m\,\ell^2}{12}\,\ddot{\theta},$$

so that

$$\ddot{\theta} = \frac{3}{m\,\ell}\left(\frac{m\,g\,\mathrm{S}_\theta}{2} - F\,\mathrm{C}_\theta\right).$$



Also, we can solve for the reactions as

$$R_A = \left(1 - \frac{3\,\mathrm{S}_\theta^2}{4}\right)\,m\,g + \frac{3\,\mathrm{S}_\theta\,\mathrm{C}_\theta}{2}\,F,$$

$$R_B = \frac{3\,\mathrm{S}_\theta\,\mathrm{C}_\theta}{4}\,m\,g + \left(1 - \frac{3\,\mathrm{C}_\theta^2}{2}\right)\,F.$$

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