Rigid Body Dynamics

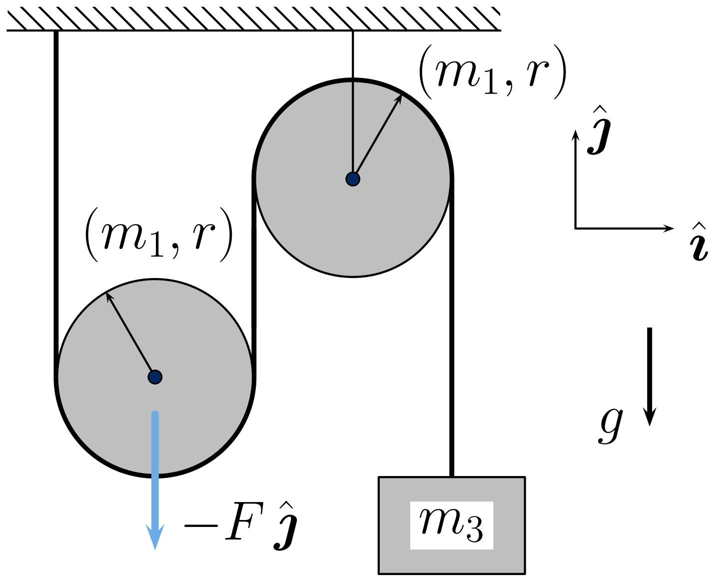
Engineering Mechanics: Dynamics

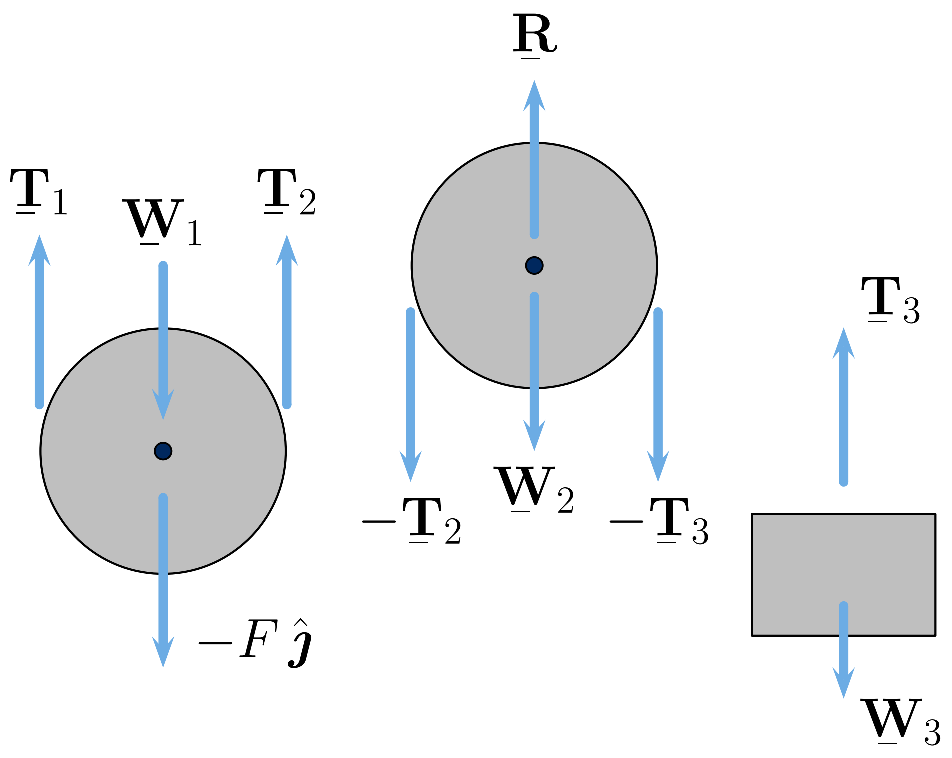
D. Dane Quinn, PhD

Department of Mechanical Engineering
The University of Akron
Akron OH 44325–3903 USA

quinn@uakron.edu
www.uakron.edu

For the pulley system shown, each pulley as mass m_1 and radius r, while the block has mass m_3. If the left pulley is subject to the force $-F\,\hat{\boldsymbol{\jmath}}$ find the resulting acceleration of the block.





The forces acting on the system arise from

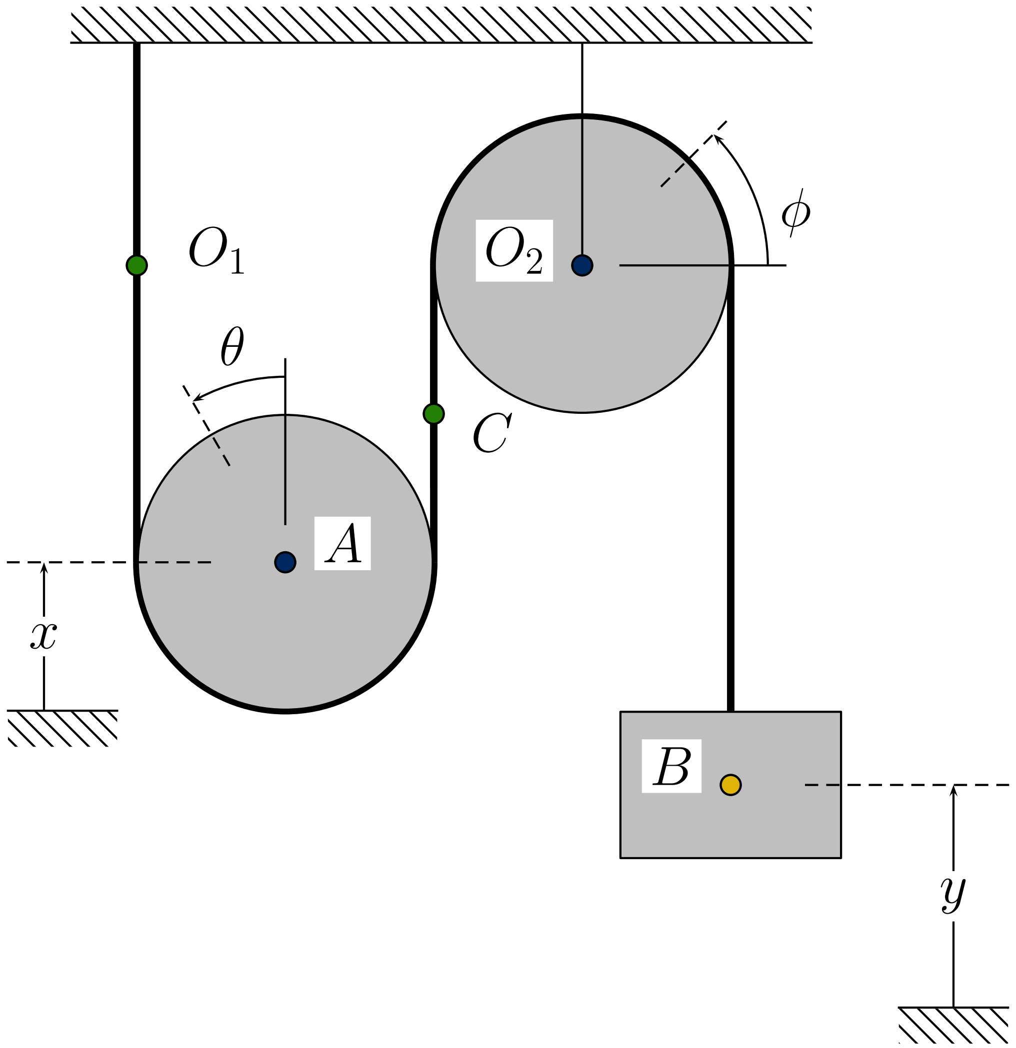
- Tension ($\underline{\mathbf{T}}_1$, $\underline{\mathbf{T}}_2$, $\underline{\mathbf{T}}_3$) in the cable
- Reaction Force ($\underline{\mathbf{R}}$)
- Weight ($\underline{\mathbf{W}}_1$, $\underline{\mathbf{W}}_2$, $\underline{\mathbf{W}}_3$) of each component
- External force ($-F\,\hat{\boldsymbol{\jmath}}$)

quinn@uakron.edu
www.uakron.edu

Coordinates and Directions/Kinematics

For each object we identify the displacement of the mass center, and additionally the rotation of each pulley, so that

$$\underline{\mathbf{a}}_A = \ddot{x}\,\hat{\boldsymbol{\jmath}}, \quad \underline{\mathbf{a}}_{O_2} = \underline{\mathbf{0}}, \quad \underline{\mathbf{a}}_B = \ddot{y}\,\hat{\boldsymbol{\jmath}},$$
$$\underline{\boldsymbol{\alpha}}_1 = \ddot{\theta}\,\hat{\boldsymbol{k}}, \quad \underline{\boldsymbol{\alpha}}_2 = \ddot{\phi}\,\hat{\boldsymbol{k}}.$$



The kinematics of the pulley system are such that

$$\underline{\mathbf{v}}_A = \frac{\underline{\mathbf{v}}_{O_1} + \underline{\mathbf{v}}_C}{2}, \quad \underline{\mathbf{v}}_{O_2} = \frac{\underline{\mathbf{v}}_C + \underline{\mathbf{v}}_B}{2},$$

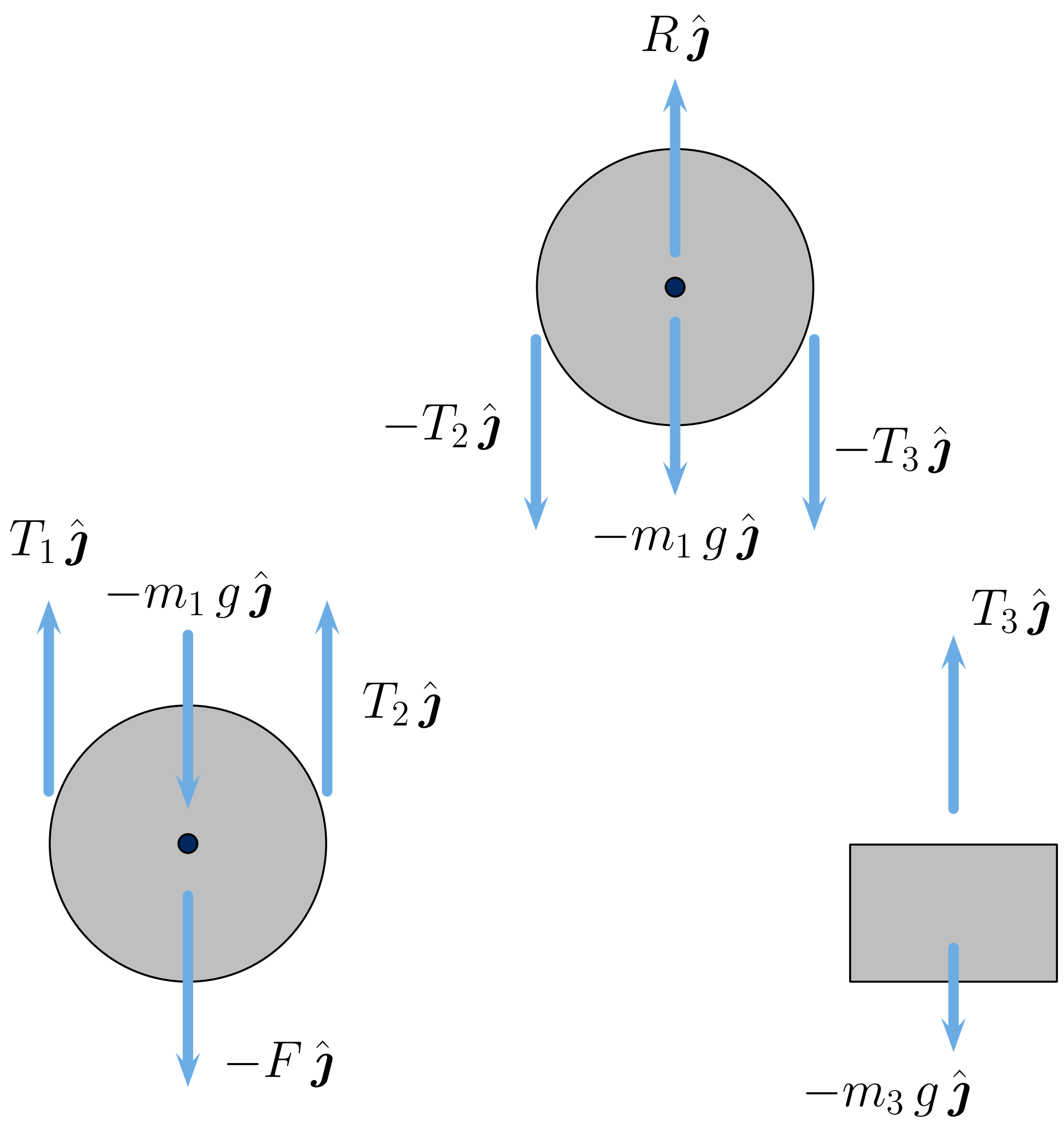
$$\longrightarrow \quad \begin{aligned} \underline{\mathbf{v}}_B &= -2\,\underline{\mathbf{v}}_A, \\ \dot{y}\,\hat{\boldsymbol{\jmath}} &= -2\,\dot{x}\,\hat{\boldsymbol{\jmath}}. \end{aligned}$$

Likewise

$$\begin{aligned} \underline{\mathbf{v}}_A &= \underline{\mathbf{v}}_{O_1} + \underline{\boldsymbol{\omega}}_1 \times \underline{\mathbf{r}}_{A/O_1}, \\ \dot{x}\,\hat{\boldsymbol{\jmath}} &= \underline{\mathbf{0}} + \left(\dot{\theta}\,\hat{\boldsymbol{k}}\right) \times (r\,\hat{\boldsymbol{\imath}}), \\ \dot{x}\,\hat{\boldsymbol{\jmath}} &= r\,\dot{\theta}\,\hat{\boldsymbol{\jmath}}, \end{aligned} \qquad \begin{aligned} \underline{\mathbf{v}}_B &= \underline{\mathbf{v}}_{O_2} + \underline{\boldsymbol{\omega}}_2 \times \underline{\mathbf{r}}_{B/O_2}, \\ \dot{y}\,\hat{\boldsymbol{\jmath}} &= \underline{\mathbf{0}} + \left(\dot{\phi}\,\hat{\boldsymbol{k}}\right) \times (r\,\hat{\boldsymbol{\imath}}), \\ \dot{y}\,\hat{\boldsymbol{\jmath}} &= r\,\dot{\phi}\,\hat{\boldsymbol{\jmath}}, \end{aligned} \quad \text{so that} \quad \begin{cases} \dot{y} = -2\,\dot{x}, \\ \dot{x} = r\,\dot{\theta}, \\ \dot{y} = r\,\dot{\phi}. \end{cases}$$

quinn@uakron.edu
www.uakron.edu

Free Body Diagram/Equations of Motion



Momentum balance for the system can be written as

Pulley 1:

$$\sum \underline{\mathbf{F}} = (T_1 + T_2 - m_1\,g - F)\,\hat{\boldsymbol{\jmath}} = m_1\,(\ddot{x}\,\hat{\boldsymbol{\jmath}}) = m_1\,\underline{\mathbf{a}}_A,$$

$$\sum \underline{\mathbf{M}}_A = (r\,T_2 - r\,T_1)\,\hat{\boldsymbol{k}} = \frac{m_1\,r^2}{2}\,\ddot{\theta}\,\hat{\boldsymbol{k}} = I_A\,\underline{\boldsymbol{\alpha}}_1,$$

Pulley 2:

$$\sum \underline{\mathbf{F}} = (R - T_2 - T_3 - m_1\,g)\,\hat{\boldsymbol{\jmath}} = \underline{\mathbf{0}} = m_1\,\underline{\mathbf{a}}_{O_2},$$

$$\sum \underline{\mathbf{M}}_{O_2} = (r\,T_2 - r\,T_3)\,\hat{\boldsymbol{k}} = \frac{m_1\,r^2}{2}\,\ddot{\phi}\,\hat{\boldsymbol{k}} = I_{O_2}\,\underline{\boldsymbol{\alpha}}_2,$$

Block:

$$\sum \underline{\mathbf{F}} = (T_3 - m_3\,g)\,\hat{\boldsymbol{\jmath}} = m_3\,(\ddot{y}\,\hat{\boldsymbol{\jmath}}) = m_3\,\underline{\mathbf{a}}_B,$$

quinn@uakron.edu
www.uakron.edu

Taking components of these equations yields

$$T_1 + T_2 - m_1\, g - F = m_1\, \ddot{x}, \qquad r\, (T_2 - T_1) = \frac{m_1\, r^2}{2}\, \ddot{\theta}, \qquad R - T_2 - T_3 - m_1\, g = 0,$$

$$r\, (T_2 - T_3) = \frac{m_1\, r^2}{2}\, \ddot{\phi}, \qquad T_3 - m_3\, g = m_3\, \ddot{y},$$

together with the constraint equations

$$\ddot{y} = -2\, \ddot{x}, \qquad \ddot{x} = r\, \ddot{\theta}, \qquad \ddot{y} = r\, \ddot{\phi}.$$

Therefore, eliminating T_1, T_2, and T_3 yields

$$m_1\, g + F + m_1\, \ddot{x} + \frac{m_1\, r}{2}\, \ddot{\theta} = 2\, \left(m_3\, g + m_3\, \ddot{y} + \frac{m_1\, r}{2}\, \ddot{\phi} \right),$$

and using the constraint equations, we solve for $\ddot{y}$ as

$$\ddot{y} = \frac{F + (m_1 - 2\, m_3)\; g}{4\, m_1 + 2\, m_3},$$

so that

$$\underline{\mathbf{a}}_B = \ddot{y}\, \hat{\boldsymbol{\jmath}} = \left(\frac{F + (m_1 - 2\, m_3)\; g}{4\, m_1 + 2\, m_3} \right)\, \hat{\boldsymbol{\jmath}}.$$

quinn@uakron.edu
www.uakron.edu