# Moment of Inertia

## Engineering Mechanics: Dynamics

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The Moment of Inertia of $\mathcal{B}$ about $A$, denoted as $\mathbf{I}_A^{\mathcal{B}}$, describes the mass distribution of $\mathcal{B}$ relative to $A$.

FROM ANGULAR MOMENTUM BALANCE

$$\int_{\mathcal{B}} \left( \underline{r}_{dm/A} \times \left( \underline{\alpha}_{\mathcal{B}} \times \underline{r}_{dm/A} \right) \right) dm \equiv \underline{\underline{I}}_A^{\mathcal{B}} \, \underline{\alpha}_{\mathcal{B}}$$

PLANAR OBJECTS

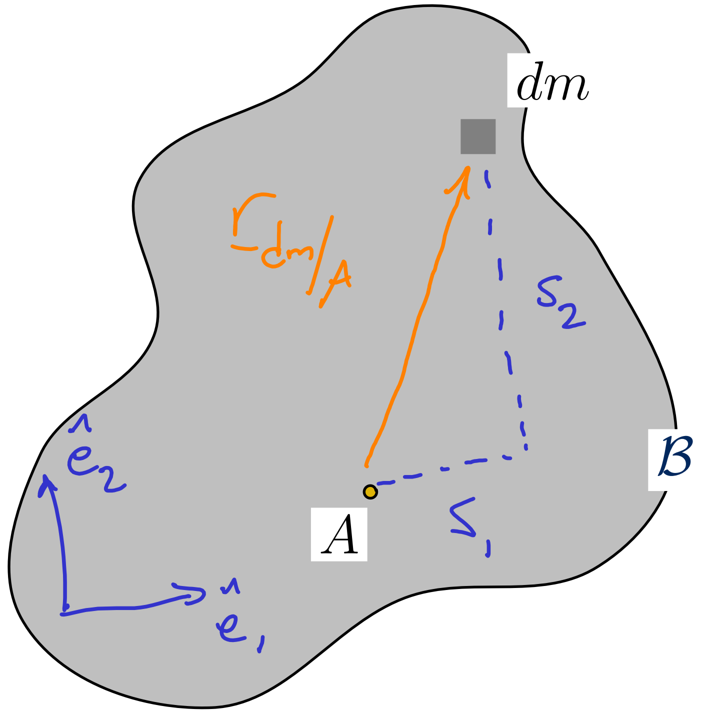
$$\underline{r}_{dm/A} = s_1 \hat{e}_1 + s_2 \hat{e}_2 \quad ; \quad \underline{\alpha}_{\mathcal{B}} = \alpha \hat{k}$$

$$\underline{r}_{dm/A} \times \left( \underline{\alpha}_{\mathcal{B}} \times \underline{r}_{dm/A} \right) = \left( s_1^2 + s_2^2 \right) \alpha \hat{k}$$

$$= \| \underline{r}_{dm/A} \|^2 \, \underline{\alpha}_{\mathcal{B}}$$

DEFINE $\| \underline{r}_{dm/A} \| \equiv s$

$$\int_{\mathcal{B}} \left( \underline{r}_{dm/A} \times \left( \underline{\alpha}_{\mathcal{B}} \times \underline{r}_{dm/A} \right) \right) dm = \left[ \int_{\mathcal{B}} s^2 \, dm \right] \underline{\alpha}_{\mathcal{B}} \equiv I_A^{\mathcal{B}} \, \underline{\alpha}_{\mathcal{B}}$$




MOMENT OF INERTIA OF $\mathcal{B}$ ABOUT THE $\hat{k}$ AXIS THROUGH $A$

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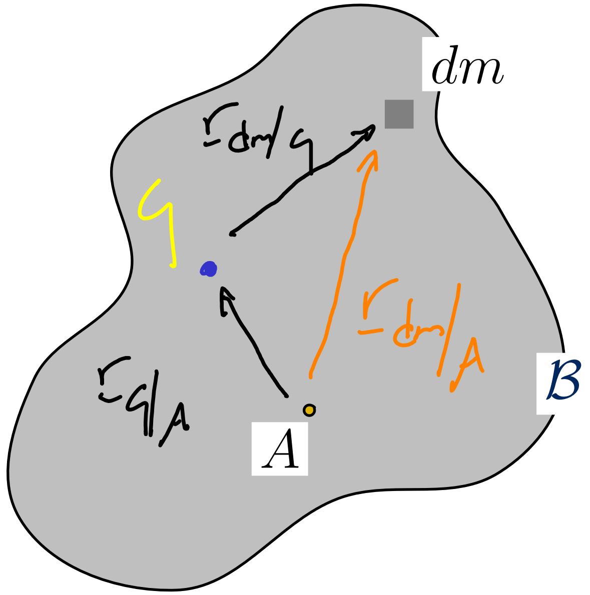
$$I_A^{\mathcal{B}}\,\underline{\alpha}_{\mathcal{B}} = \int_{\mathcal{B}} \left(\underline{r}_{dm/A} \times \left(\underline{\alpha}_{\mathcal{B}} \times \underline{r}_{dm/A}\right)\right) dm$$

$$\underline{r}_{dm/A} = \underline{r}_{G/A} + \underline{r}_{dm/G}$$

$$= \int_{\mathcal{B}} \left(\underline{r}_{G/A} \times \left(\underline{\alpha}_{\mathcal{B}} \times \underline{r}_{G/A}\right)\right) dm + \int_{\mathcal{B}} \left(\cancelto{\underline{0}}{\underline{r}_{G/A} \times \left(\underline{\alpha}_{\mathcal{B}} \times \underline{r}_{dm/G}\right)}\right) dm$$

$$+ \int_{\mathcal{B}} \left(\cancelto{\underline{0}}{\underline{r}_{dm/G} \times} \left(\underline{\alpha}_{\mathcal{B}} \times \underline{r}_{G/A}\right)\right) dm + \int_{\mathcal{B}} \left(\underline{r}_{dm/G} \times \left(\underline{\alpha}_{\mathcal{B}} \times \underline{r}_{dm/G}\right)\right) dm$$




$$\int_{\mathcal{B}} \underline{r}_{dm/G} = \underline{0}$$

$$I_A^{\mathcal{B}}\,\underline{\alpha}_{\mathcal{B}} = \underbrace{\int_{\mathcal{B}} \left(\underline{r}_{G/A} \times \left(\underline{\alpha}_{\mathcal{B}} \times \underline{r}_{G/A}\right)\right) dm}_{m_{\mathcal{B}} \left\| \underline{r}_{G/A} \right\|^2 \underline{\alpha}_{\mathcal{B}}} + \underbrace{\int_{\mathcal{B}} \left(\underline{r}_{dm/G} \times \left(\underline{\alpha}_{\mathcal{B}} \times \underline{r}_{dm/G}\right)\right) dm}_{I_G^{\mathcal{B}}\,\underline{\alpha}_{\mathcal{B}}}$$

$$I_A^{\mathcal{B}}\,\underline{\alpha}_{\mathcal{B}} = \left(I_G^{\mathcal{B}} + m \left\| \underline{r}_{G/A} \right\|^2\right) \underline{\alpha}_{\mathcal{B}}$$

$$I_A^{\mathcal{B}} = I_G^{\mathcal{B}} + m d^2 \qquad d = \left\| \underline{r}_{G/A} \right\|$$

PARALLEL-AXIS THEOREM

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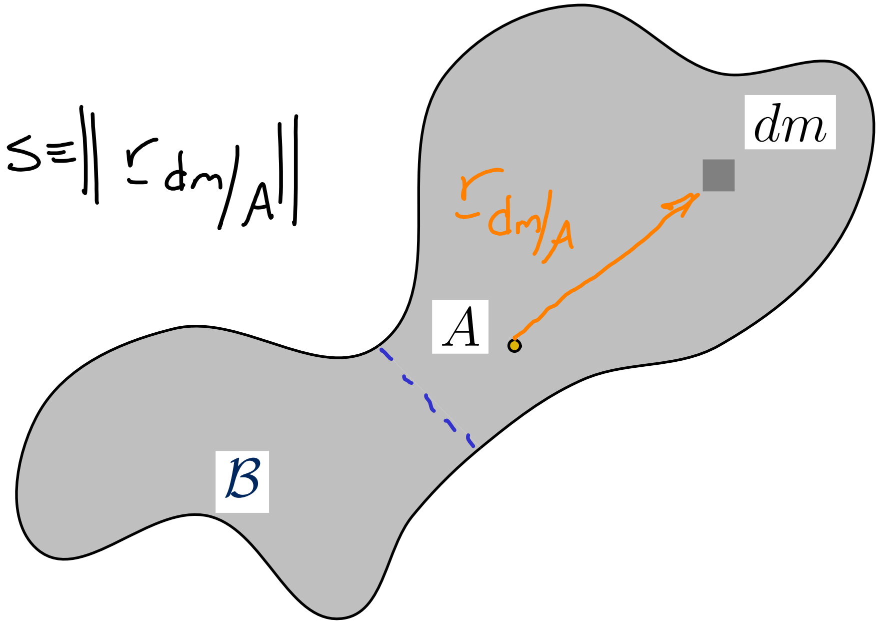
# Composite objects

$$I_A^{\mathcal{B}} = \int_{\mathcal{B}} s^2\,dm = \int_{\mathcal{B}_1} s^2\,dm + \int_{\mathcal{B}_2} s^2\,dm$$

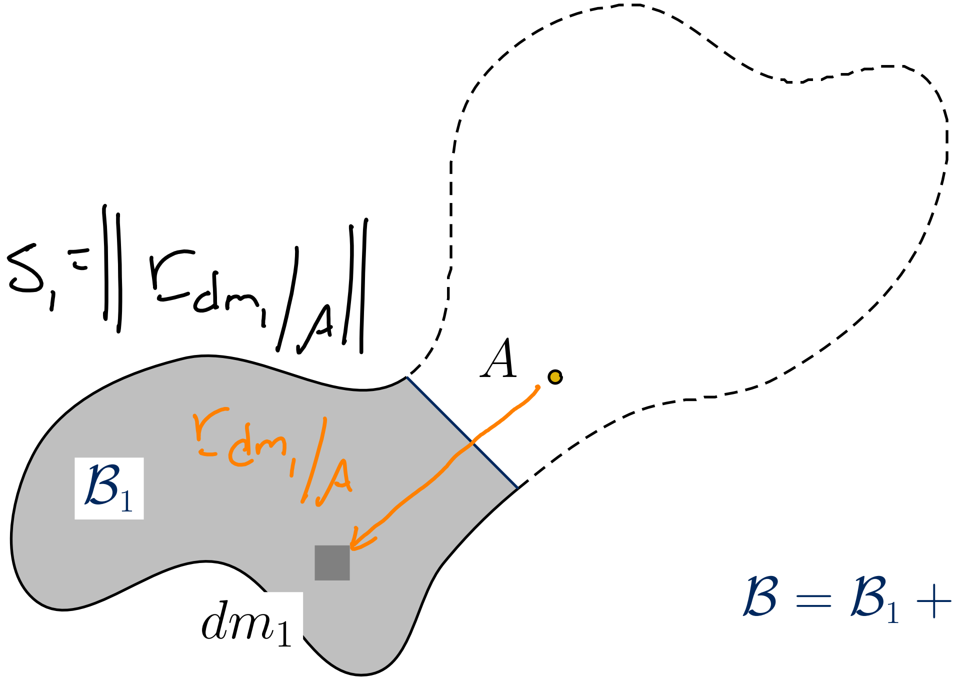
$$= \int_{\mathcal{B}_1} s_1^2\,dm_1 + \int_{\mathcal{B}_2} s_2^2\,dm$$

$$= I_A^{\mathcal{B}_1} + I_A^{\mathcal{B}_2}$$

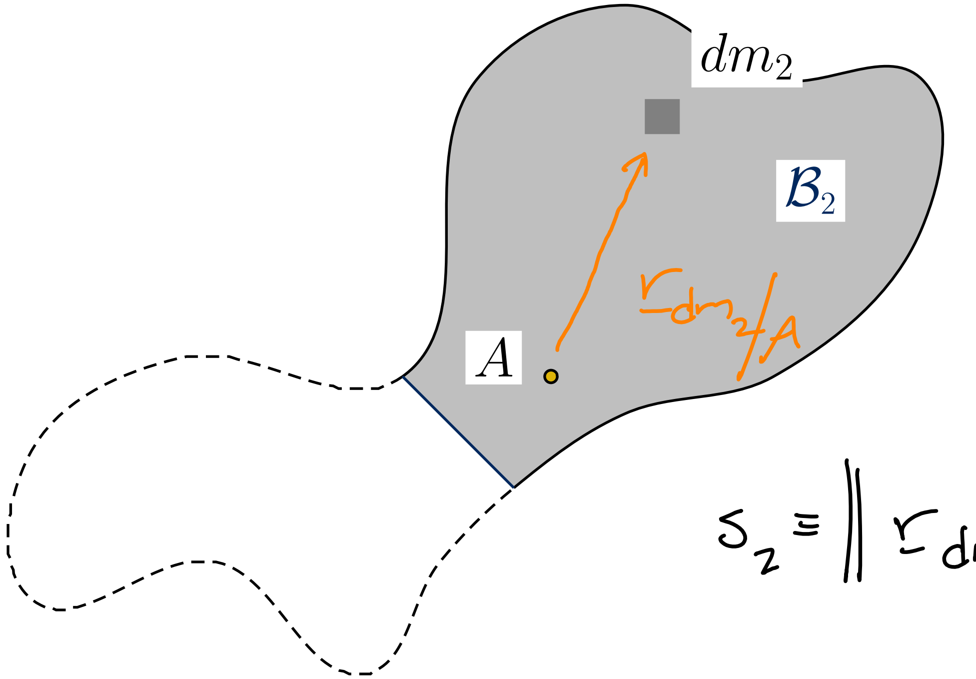
$s \equiv \| \underline{r}_{dm/A} \|$




$s_1 = \| \underline{r}_{dm_1/A} \|$




$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$




$s_2 \equiv \| \underline{r}_{dm_2/A} \|$

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# Radius of Gyration

RADIUS OF GYRATION $\left( k_A^B \right)$ IS DEFINED AS

$$k_A^B = \sqrt{\frac{I_A^B}{m_B}} \qquad \boxed{I_A^B = m_B \left( k_A^B \right)^2}$$

- UNITS OF LENGTH
- FOR OBJECTS WITH UNIFORM DENSITY, $k_A^B$ IS INDEPENDENT OF $\rho$

$$\rho = \frac{m_B}{V_B} \longrightarrow \left( dm = \rho\, dV \right)$$

$$\left( k_A^B \right)^2 = \frac{\int_B s^2 \cancel{\rho}\, dV}{\int_B \cancel{\rho}\, dV} = \frac{\int_B s^2\, dV}{V_B}$$

DEPENDS ON THE SHAPE BUT NOT THE MATERIAL

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