Moment of Inertia

Engineering Mechanics: Dynamics

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The Moment of Inertia of $\mathcal{B}$ about A, denoted as $\mathbf{I}_A^{\mathcal{B}}$, describes the mass distribution of $\mathcal{B}$ relative to A.

From angular momentum balance

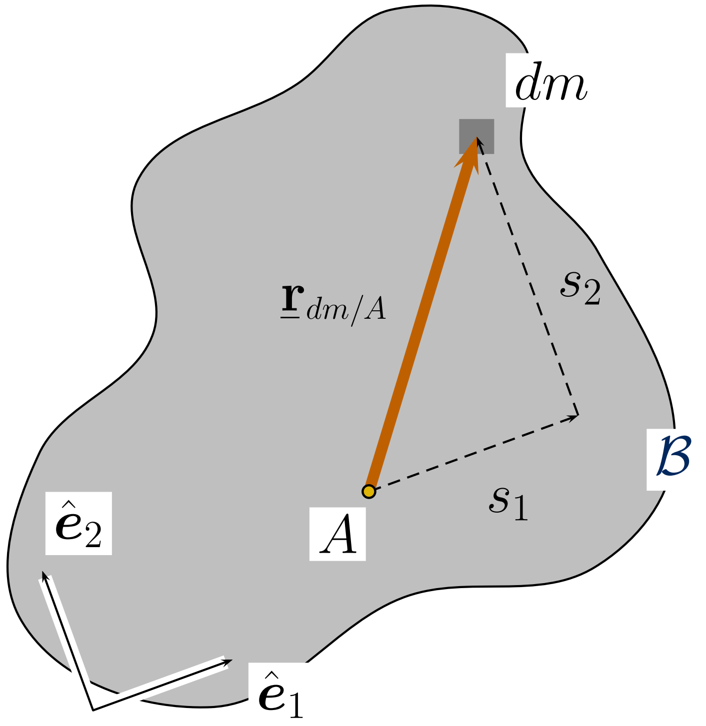
$$\int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{dm/A} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A}\right)\right) \, dm \equiv \mathbf{I}_A^{\mathcal{B}} \, \underline{\boldsymbol{\alpha}}_{\mathcal{B}}.$$

For *planar* objects

$$\underline{\mathbf{r}}_{dm/A} = s_1 \, \hat{\boldsymbol{e}}_1 + s_2 \, \hat{\boldsymbol{e}}_2,$$

so that, with $\underline{\boldsymbol{\alpha}}_{\mathcal{B}} = \alpha \, \hat{\boldsymbol{k}}$

$$\begin{aligned} \underline{\mathbf{r}}_{dm/A} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A}\right) &= \left(s_1^2 + s_2^2\right) \, \alpha \, \hat{\boldsymbol{k}}, \\ &= \left\|\underline{\mathbf{r}}_{dm/A}\right\|^2 \underline{\boldsymbol{\alpha}}_{\mathcal{B}}. \end{aligned}$$



With $\left\|\underline{\mathbf{r}}_{dm/A}\right\| \equiv s$,

$$\begin{aligned} \int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{dm/A} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A}\right)\right) \, dm &= \left\{\int_{\mathcal{B}} s^2 \, dm\right\} \underline{\boldsymbol{\alpha}}_{\mathcal{B}}, \\ &\equiv \underbrace{I_A^{\mathcal{B}}} \, \underline{\boldsymbol{\alpha}}_{\mathcal{B}}. \end{aligned}$$

$I_A^{\mathcal{B}}$: Moment of Inertia of $\mathcal{B}$ about the $\hat{\boldsymbol{k}}$ axis through A

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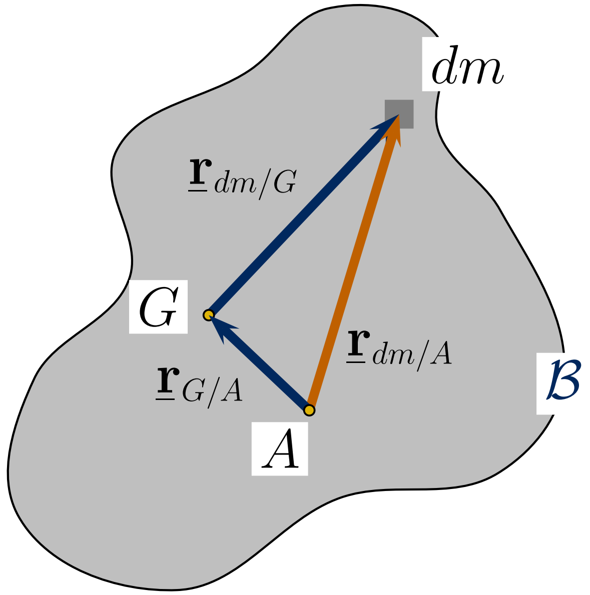
$$I_A^{\mathcal{B}}\,\underline{\boldsymbol{\alpha}}_{\mathcal{B}} = \int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{dm/A} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A}\right)\right)\,dm,$$

with $\underline{\mathbf{r}}_{dm/A} = \underline{\mathbf{r}}_{G/A} + \underline{\mathbf{r}}_{dm/G}$

$$\begin{aligned} &= \int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{dm/G} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/G}\right)\right)\,dm \\ &\quad + \int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{G/A} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/G}\right)\right)\,dm + \int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{dm/G} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{G/A}\right)\right)\,dm \\ &\quad + \int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{G/A} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{G/A}\right)\right)\,dm. \end{aligned}$$



Since G is the mass center, $\int_{\mathcal{B}} \underline{\mathbf{r}}_{dm/G}\,dm = \underline{\mathbf{0}}$, and

$$I_A^{\mathcal{B}}\,\underline{\boldsymbol{\alpha}}_{\mathcal{B}} = \underbrace{\int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{dm/G} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/G}\right)\right)\,dm}_{I_G^{\mathcal{B}}\,\underline{\boldsymbol{\alpha}}_{\mathcal{B}}} + \underbrace{\int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{G/A} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{G/A}\right)\right)\,dm}_{m_{\mathcal{B}}\,\|\underline{\mathbf{r}}_{G/A}\|^2\,\underline{\boldsymbol{\alpha}}_{\mathcal{B}}},$$

so that

$$I_A^{\mathcal{B}}\,\underline{\boldsymbol{\alpha}}_{\mathcal{B}} = I_G^{\mathcal{B}}\,\underline{\boldsymbol{\alpha}}_{\mathcal{B}} + m_{\mathcal{B}}\,\|\underline{\mathbf{r}}_{G/A}\|^2\,\underline{\boldsymbol{\alpha}}_{\mathcal{B}}, \quad \longrightarrow \quad I_A^{\mathcal{B}} = I_G^{\mathcal{B}} + m_{\mathcal{B}}\,\|\underline{\mathbf{r}}_{G/A}\|^2$$

Parallel-axis theorem

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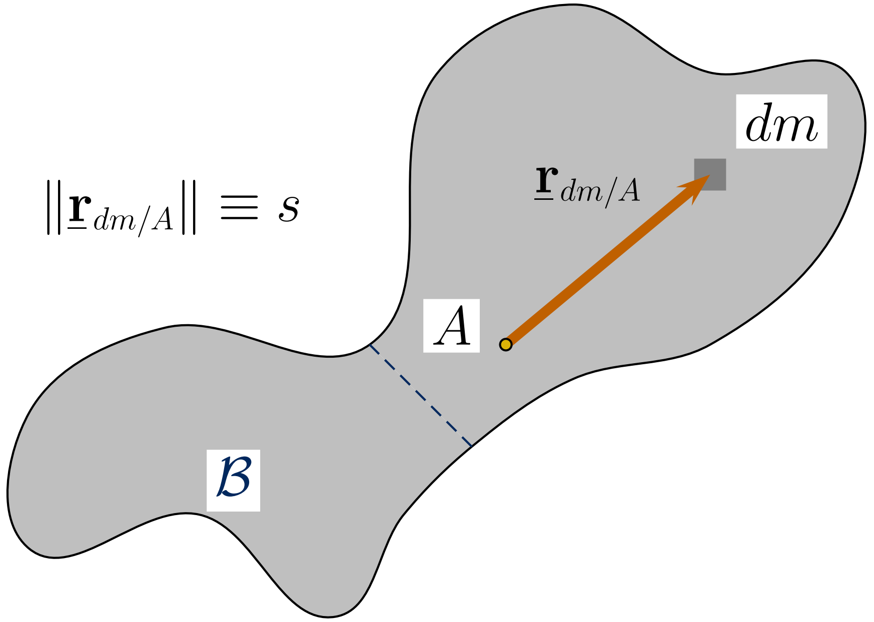
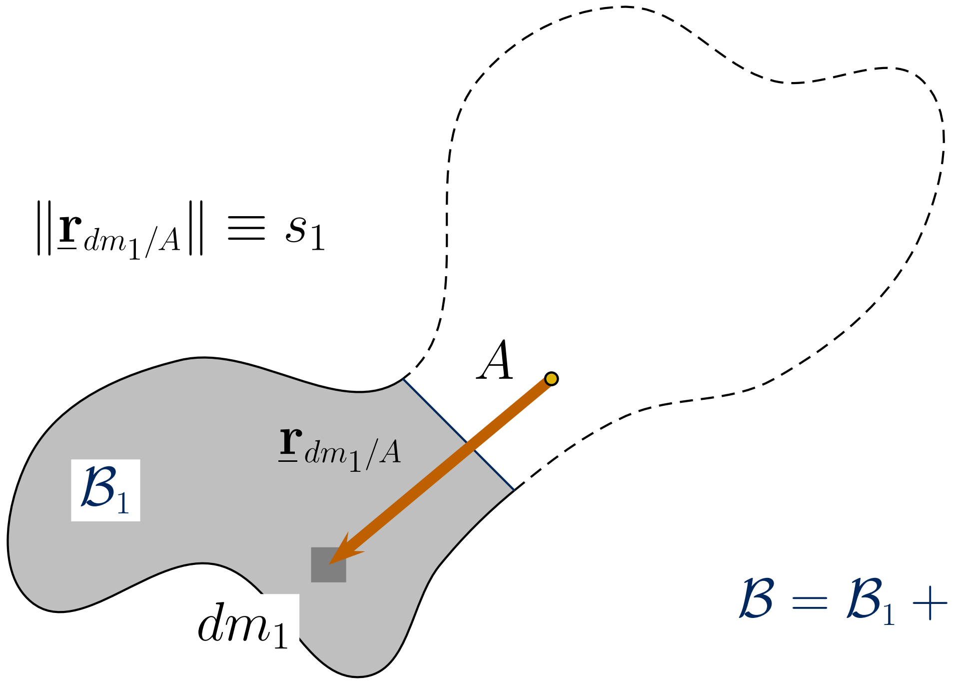
Composite objects

If an object $\mathcal{B}$ is decomposed into two (or more) components, so that $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$, then

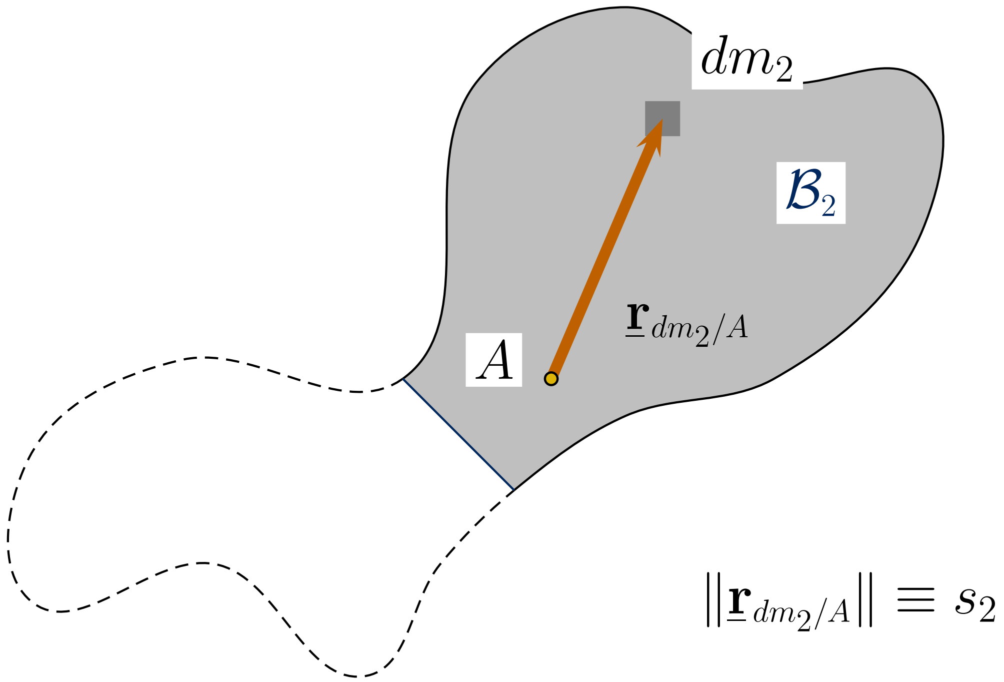
$$I_A^{\mathcal{B}} = \int_{\mathcal{B}} s^2\, dm = \int_{\mathcal{B}_1} s_1^2\, dm_1 + \int_{\mathcal{B}_2} s_2^2\, dm_2,$$
$$= I_A^{\mathcal{B}_1} + I_A^{\mathcal{B}_2}.$$

Notice that A need not be physically on the body





$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$



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Radius of Gyration

The radius of gyration ($k_A^{\mathcal{B}}$) of an object $\mathcal{B}$ about a point A is defined as

$$k_A^{\mathcal{B}} \equiv \sqrt{\frac{I_A^{\mathcal{B}}}{m_{\mathcal{B}}}},$$

so that $I_A^{\mathcal{B}} = m_{\mathcal{B}} \left(k_A^{\mathcal{B}}\right)^2$.

Characteristics of $k_A^{\mathcal{B}}$:

- has units of length
- for an object of uniform density ρ, $k_A^{\mathcal{B}}$ is independent of ρ

$$\rho = \frac{m_{\mathcal{B}}}{V_{\mathcal{B}}}, \ (dm = \rho\, dV) \quad \longrightarrow \quad \left(k_A^{\mathcal{B}}\right)^2 = \frac{\int_{\mathcal{B}} s^2 \rho\, dV}{\int_{\mathcal{B}} \rho\, dV} = \frac{\int_{\mathcal{B}} s^2\, dV}{V_{\mathcal{B}}},$$

$k_A^{\mathcal{B}}$ depends on the shape of $\mathcal{B}$ but not the material

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