Momentum Balance

Engineering Mechanics: Dynamics

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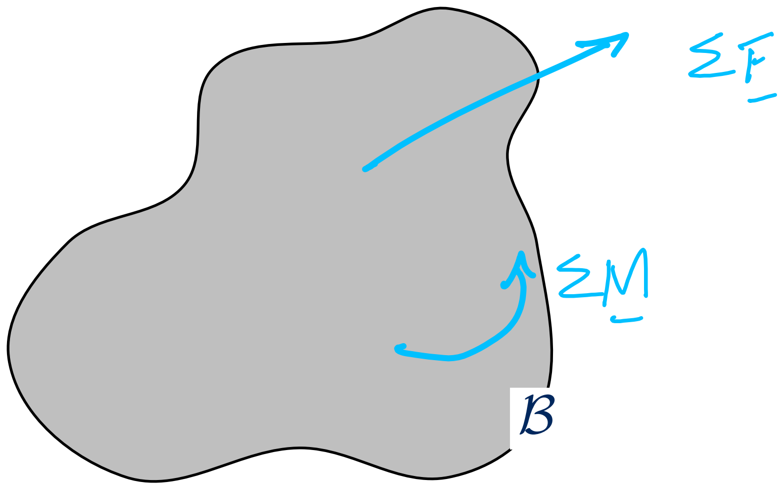
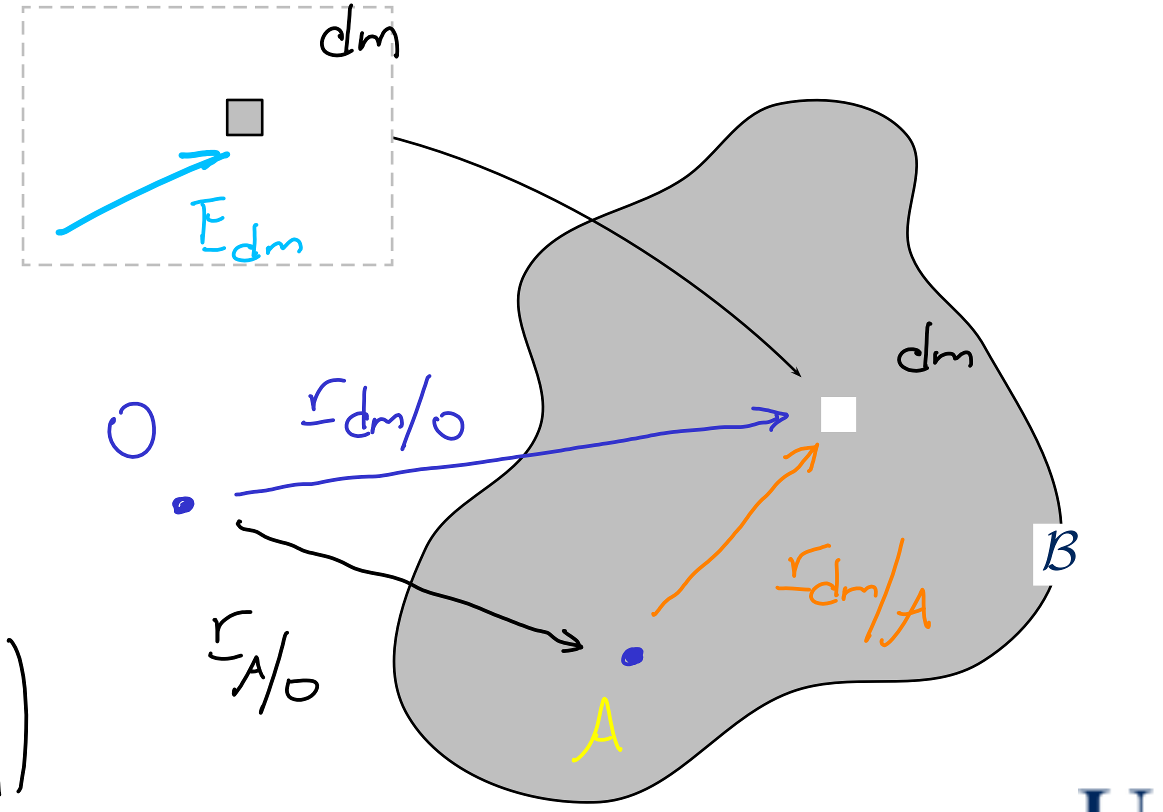
Dynamics: Application of the laws of mechanics to develop equations of motion that describe the response of a system to external effects



- Relate the external forces ($\underline{\mathbf{F}}$) and moments ($\underline{\mathbf{M}}$) acting on a rigid object ($\mathcal{B}$) to the motion of that object
- Planar motion—all points are constrained to move in a single plane of motion

Linear Momentum Balance

$$\underline{F}_{dm} = dm\, \underline{a}_{dm}$$



ACCELERATION

$$\underline{a}_{dm} = \frac{d^2}{dt^2}\left(\underline{r}_{dm/O}\right) = \underline{a}_A + \underline{\alpha}_{\mathcal{B}} \times \underline{r}_{dm/A} + \underline{\omega}_{\mathcal{B}} \times \left(\underline{\omega}_{\mathcal{B}} \times \underline{r}_{dm/A}\right)$$

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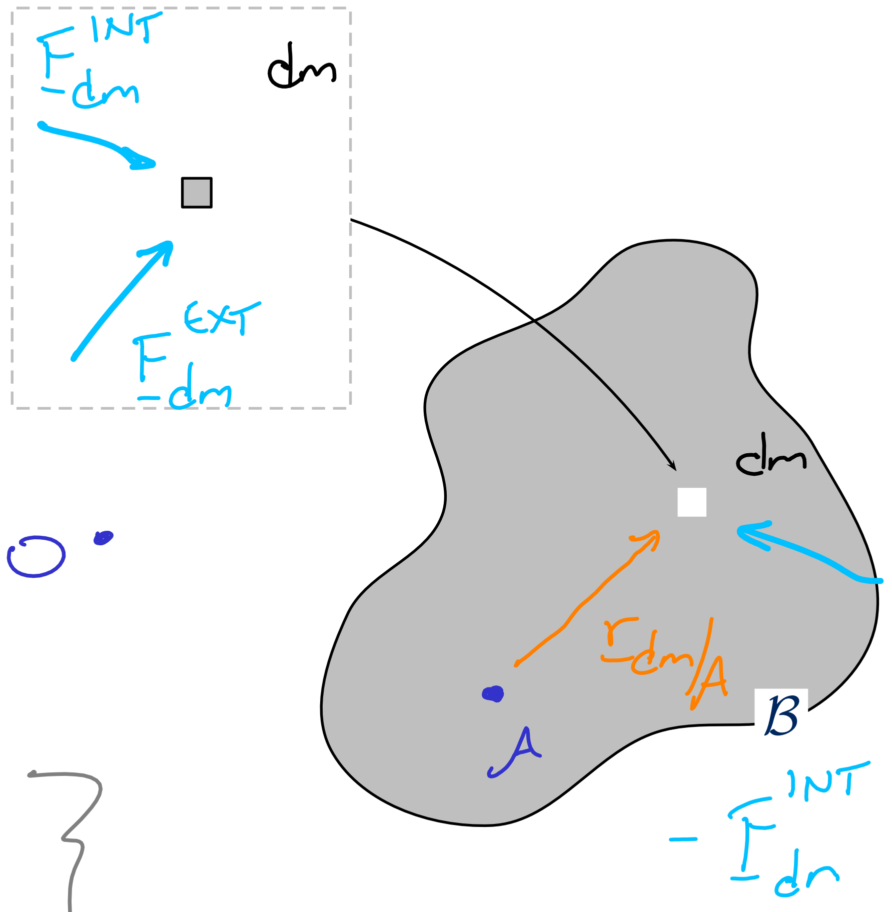
TOTAL FORCE $\underline{F}_{dm}$

$$\underline{F}_{dm} = \underline{F}^{EXT}_{dm} + \underline{F}^{INT}_{dm}$$

$$\int_{\mathcal{B}} \left\{ \underline{F}_{dm} = dm\, \underline{a}_{dm} \right\}$$

$$\int_{\mathcal{B}} \left\{ \underline{F}^{EXT}_{dm} + \underline{F}^{INT}_{dm} = \left(\underline{a}_A + \underline{\alpha}_{\mathcal{B}} \times \underline{r}_{dm/A} + \underline{\omega}_{\mathcal{B}} \times \left(\underline{\omega}_{\mathcal{B}} \times \underline{r}_{dm/A} \right) \right) dm \right\}$$

$$\int_{\mathcal{B}} \underline{F}^{EXT}_{dm} + \int_{\mathcal{B}} \underline{F}^{INT}_{dm} = \underline{a}_A \int_{\mathcal{B}} dm + \underline{\alpha}_{\mathcal{B}} \times \int_{\mathcal{B}} \underline{r}_{dm/A}\, dm + \underline{\omega}_{\mathcal{B}} \times \left(\underline{\omega}_{\mathcal{B}} \times \int_{\mathcal{B}} \underline{r}_{dm/A}\, dm \right)$$



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CHOOSE A

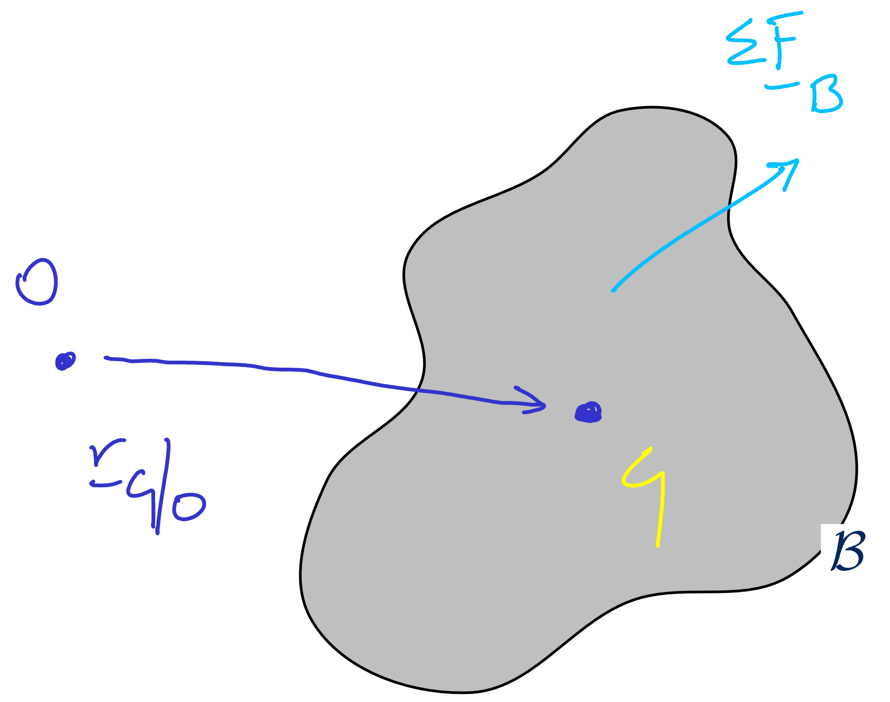
$$\int_{\mathcal{B}} \underline{r}_{dm/A}\, dm = \underline{0} \longrightarrow$$ A IS THE MASS CENTER $(A \equiv G)$

$$\int_{\mathcal{B}} \underline{F}^{INT}_{dm} = \underline{0}$$ NET INTERNAL FORCE VANISHES

$$\int_{\mathcal{B}} \underline{F}^{EXT}_{dm} = \Sigma \underline{F}_{\mathcal{B}}$$ TOTAL EXTERNAL FORCE

$$\int_{\mathcal{B}} \underline{F}^{EXT}_{dm} + \cancel{\int_{\mathcal{B}} \underline{F}^{INT}_{dm}}^{\,0} = \underline{a}_G \int_{\mathcal{B}} dm + \underline{\alpha}_{\mathcal{B}} \times \cancel{\int_{\mathcal{B}} \underline{r}_{dm/G}}^{\,0}\, dm$$

$$+ \underline{\omega}_{\mathcal{B}} \times \left(\underline{\omega}_{\mathcal{B}} \times \cancel{\int_{\mathcal{B}} \underline{r}_{dm/G}}^{\,0}\, dm \right)$$



& WE ARE LEFT WITH

$$\boxed{\Sigma \underline{F}_{\mathcal{B}} = m_{\mathcal{B}}\, \underline{a}_G}$$

$$\int_{\mathcal{B}} dm = m_{\mathcal{B}}$$

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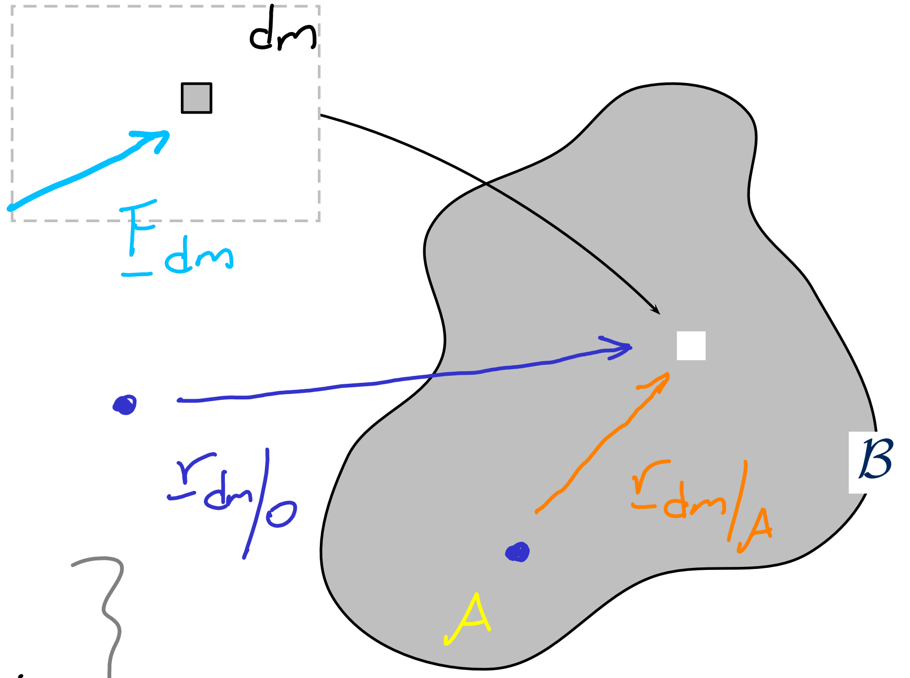
Angular Momentum Balance



$$\underline{r}_{dm/A} \times \left\{ \underline{F}_{dm} = dm\, \underline{a}_{dm} \right\}$$

$$\underline{F}_{dm} = \underline{F}^{EXT}_{dm} + \underline{F}^{INT}_{dm}$$

$$\int_B \left\{ \underline{r}_{dm/A} \times \left(\underline{F}^{EXT}_{dm} + \underline{F}^{INT}_{dm} \right) = \underline{r}_{dm/A} \times \left(\underline{a}_A + \underline{\alpha}_\beta \times \underline{r}_{dm/A} + \underline{\omega}_\beta \times \left(\underline{\omega}_\beta \times \underline{r}_{dm/A} \right) \right) dm \right\}$$

$$\int_B \left(\underline{r}_{dm/A} \times \underline{F}^{EXT}_{dm} \right) dm = \sum \underline{M}_A \qquad \int_B \left(\underline{r}_{dm/A} \times \underline{F}^{INT}_{dm} \right) dm = \underline{0}$$

FOR PLANAR OBJECTS

$$\underline{r}_{dm/A} \times \left(\underline{\omega}_\beta \times \left(\underline{\omega}_\beta \times \underline{r}_{dm/A} \right) \right) = \underline{0}$$

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ANGULAR MOMENTUM BALANCE

$$\sum \underline{M}_A^{\mathcal{B}} = \int_{\mathcal{B}} \left(\underline{r}_{dm/A} \times \underline{a}_A\right) dm + \int_{\mathcal{B}} \left(\underline{r}_{dm/A} \times \left(\underline{\alpha}_{\beta} \times \underline{r}_{dm/A}\right)\right) dm$$

& LET

$$\underline{r}_{dm/A} = \underline{r}_{G/A} + \underline{r}_{dm/G} \qquad \int_{\mathcal{B}} \underline{r}_{dm/G} = \underline{0}$$

$$\sum \underline{M}_A = \underline{r}_{G/A} \times m_{\mathcal{B}} \underline{a}_A + \int_{\mathcal{B}} \underline{r}_{dm/A} \times \left(\underline{\alpha}_{\beta} \times \underline{r}_{dm/A}\right) dm$$

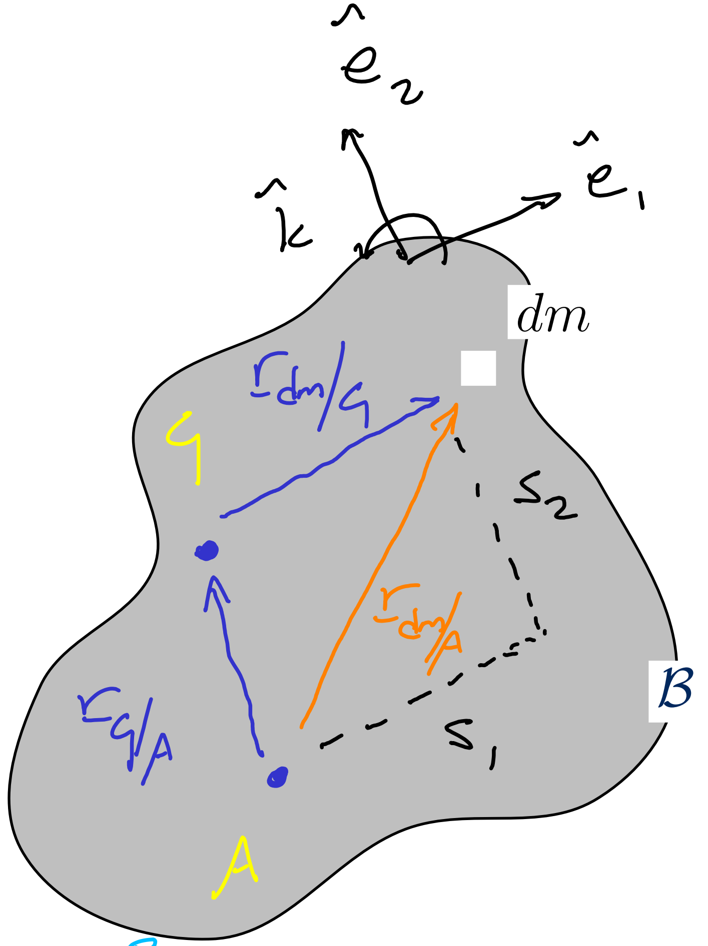
LET

$$\underline{r}_{dm/A} = s_1 \hat{e}_1 + s_2 \hat{e}_2 \qquad \& \qquad \underline{\alpha}_{\beta} = \alpha \hat{k}$$

$$\underline{r}_{dm/A} \times \left(\underline{\alpha}_{\beta} \times \underline{r}_{dm/A}\right) = \left(s_1^2 + s_2^2\right) \alpha \hat{k} = \left\|\underline{r}_{dm/A}\right\|^2 \underline{\alpha}_{\beta}$$

DEFINE $\left\|\underline{r}_{dm/A}\right\| \equiv s$

$$\int_{\mathcal{B}} \underline{r}_{dm/A} \times \left(\underline{\alpha}_{\beta} \times \underline{r}_{dm/A}\right) dm = \int_{\mathcal{B}} s^2 \underline{\alpha}_{\beta}\, dm = \left[\int_{\mathcal{B}} s^2\, dm\right] \underline{\alpha}_{\beta}$$



$I_A^{\mathcal{B}}$: MOMENT OF INERTIA OF $\mathcal{B}$ ABOUT A

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FOR PLANAR OBJECTS

$$\sum \underline{M}_A^B = \underline{r}_{g/A} \times m\underline{a}_A + I_A^B \underline{\alpha}_B$$

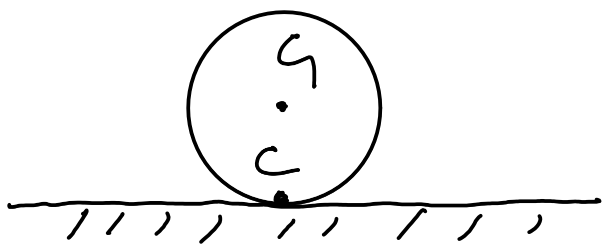
I. A IS THE MASS CENTER $\left(\underline{r}_{g/A} = \underline{0},\ A = g\right)$

$$\sum \underline{M}_g^B = I_g^B \underline{\alpha}_B$$

II. A IS FIXED IN B & FIXED IN GROUND $\left(\underline{a}_A = 0,\ A = O\right)$

$$\sum \underline{M}_O^B = I_O^B \underline{\alpha}_B$$

III. $\underline{r}_{g/A} \parallel \underline{a}_A$



DISK ROLLING WITHOUT SLIP ON GROUND

$$\sum \underline{M}_C^B = I_C^B \underline{\alpha}_B$$

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