Momentum Balance

Engineering Mechanics: Dynamics

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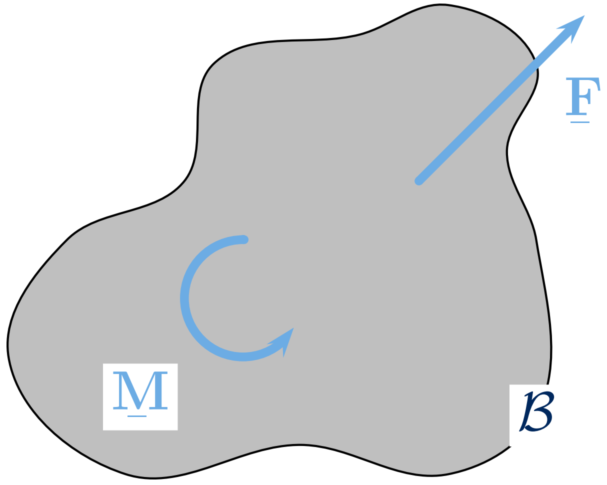
Dynamics: Application of the laws of mechanics to develop equations of motion that describe the response of a system to external effects



- Relate the external forces ($\underline{\mathbf{F}}$) and moments ($\underline{\mathbf{M}}$) acting on a rigid object ($\mathcal{B}$) to the motion of that object
- Planar motion—all points are constrained to move in a single plane of motion

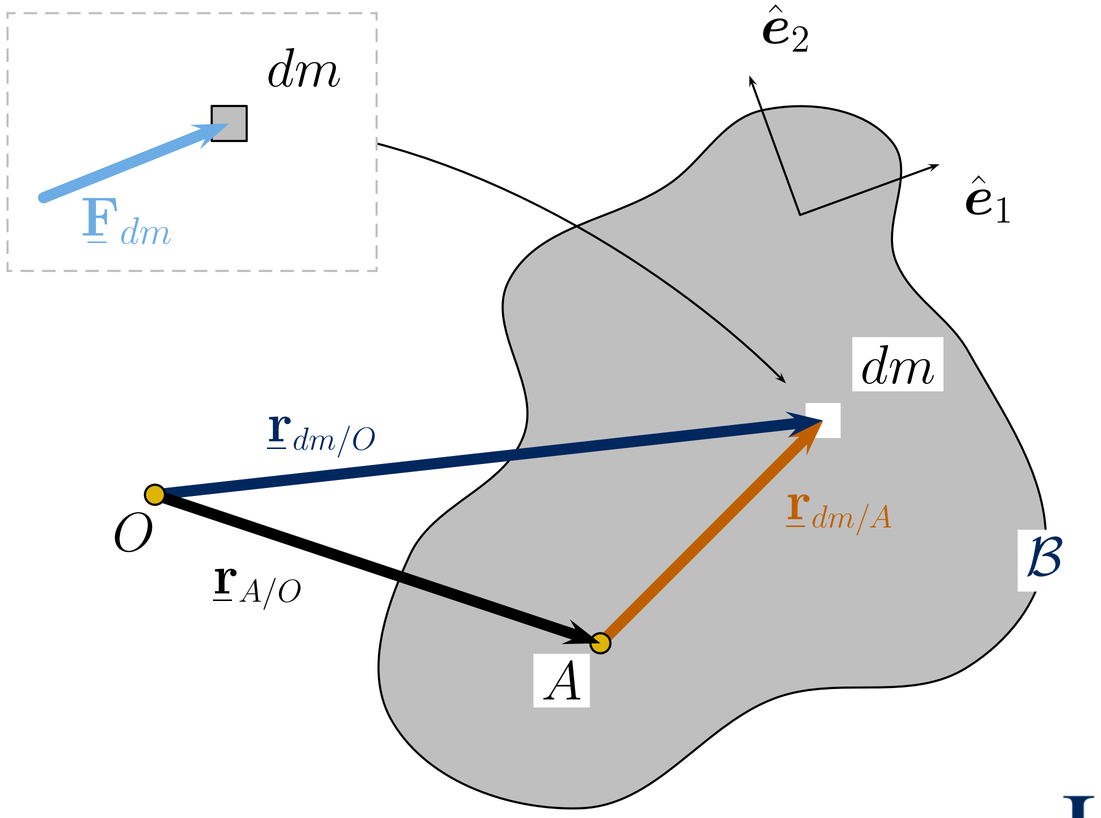
Linear Momentum Balance

Holds for each mass element dm

$$\underline{\mathbf{F}}_{dm} = (dm)\,\underline{\mathbf{a}}_{dm}$$

The acceleration of each mass element $\underline{\mathbf{a}}_{dm}$ is described in terms of the motion of A and the rotation of $\mathcal{B}$

$$\underline{\mathbf{a}}_{dm} = \frac{d^2}{dt^2}\left(\underline{\mathbf{r}}_{dm/O}\right) = \underline{\mathbf{a}}_A + \underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A} + \underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \left(\underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A}\right).$$



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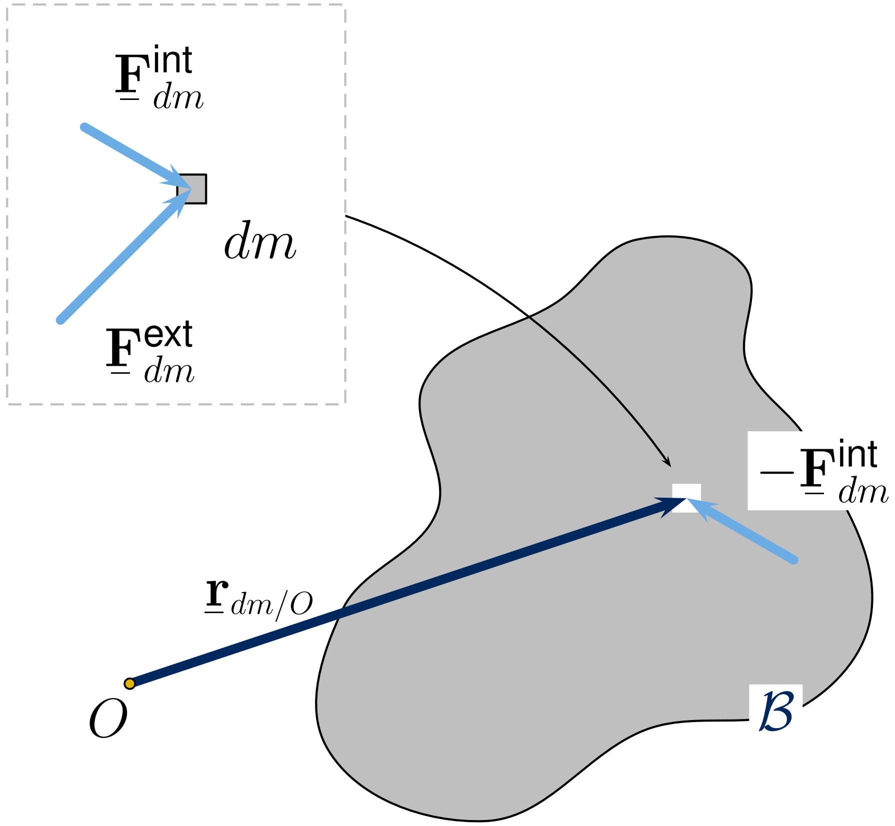
The total force acting on dm can be split into a component arising from forces external to $\mathcal{B}$ and an internal component (stress)

$$\underline{\mathbf{F}}_{dm} = \underline{\mathbf{F}}_{dm}^{\text{external}} + \underline{\mathbf{F}}_{dm}^{\text{internal}}$$

Linear momentum balance for each mass element is then integrated over the body

$$\int_{\mathcal{B}} \left\{ \underline{\mathbf{F}}_{dm} = \underline{\mathbf{a}}_{dm}\, dm \right\}$$



$$\int_{\mathcal{B}} \left\{ \underline{\mathbf{F}}_{dm}^{\text{external}} + \underline{\mathbf{F}}_{dm}^{\text{internal}} = \left(\underline{\mathbf{a}}_A + \underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A} + \underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \left(\underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A}\right)\right)\, dm \right\},$$

$$\int_{\mathcal{B}} \underline{\mathbf{F}}_{dm}^{\text{external}} + \int_{\mathcal{B}} \underline{\mathbf{F}}_{dm}^{\text{internal}} = \underline{\mathbf{a}}_A \int_{\mathcal{B}} dm + \underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \int_{\mathcal{B}} \underline{\mathbf{r}}_{dm/A}\, dm + \underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \left(\underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \int_{\mathcal{B}} \underline{\mathbf{r}}_{dm/A}\, dm\right)$$

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Choose A so that

$$\int_{\mathcal{B}} \underline{\mathbf{r}}_{dm/A}\, dm = \mathbf{0}, \quad \longrightarrow \quad A \text{ is the mass center of } \mathcal{B}\ (\equiv G)$$

Also

$$\int_{\mathcal{B}} \underline{\mathbf{F}}_{dm}^{\text{internal}} = \mathbf{0}, \qquad \text{Net internal force vanishes,}$$

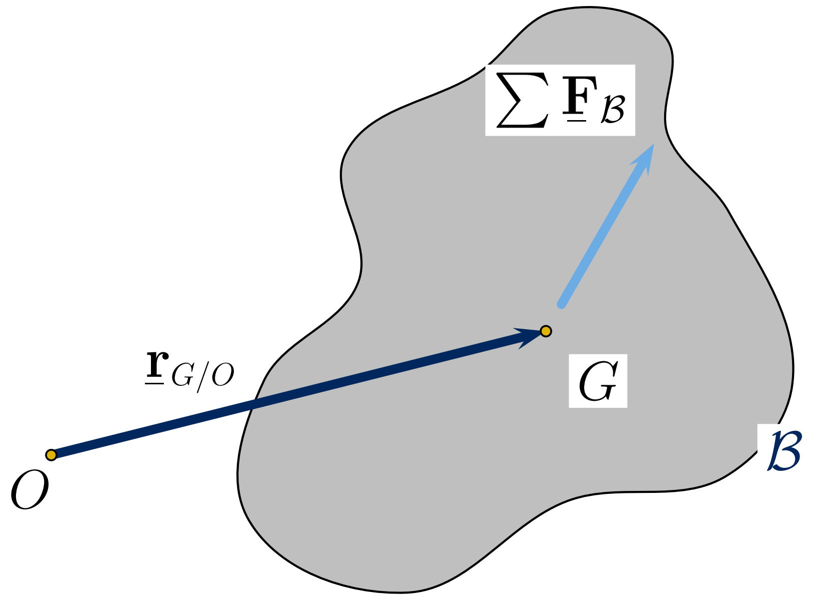
$$\int_{\mathcal{B}} \underline{\mathbf{F}}_{dm}^{\text{external}} = \sum \underline{\mathbf{F}}_{\mathcal{B}}, \qquad \text{Total external forces acting on } \mathcal{B},$$

As a result Linear Momentum Balance

$$\int_{\mathcal{B}} \underline{\mathbf{F}}_{dm}^{\text{external}} + \int_{\mathcal{B}} \underline{\mathbf{F}}_{dm}^{\text{internal}} = \underline{\mathbf{a}}_G \int_{\mathcal{B}} dm + \underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \int_{\mathcal{B}} \underline{\mathbf{r}}_{dm/G}\, dm + \underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \left(\underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \int_{\mathcal{B}} \underline{\mathbf{r}}_{dm/G}\, dm \right),$$

reduces to

$$\sum \underline{\mathbf{F}}_{\mathcal{B}} = m_{\mathcal{B}}\, \underline{\mathbf{a}}_G, \qquad \int_{\mathcal{B}} dm \equiv m_{\mathcal{B}}.$$



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Angular Momentum Balance

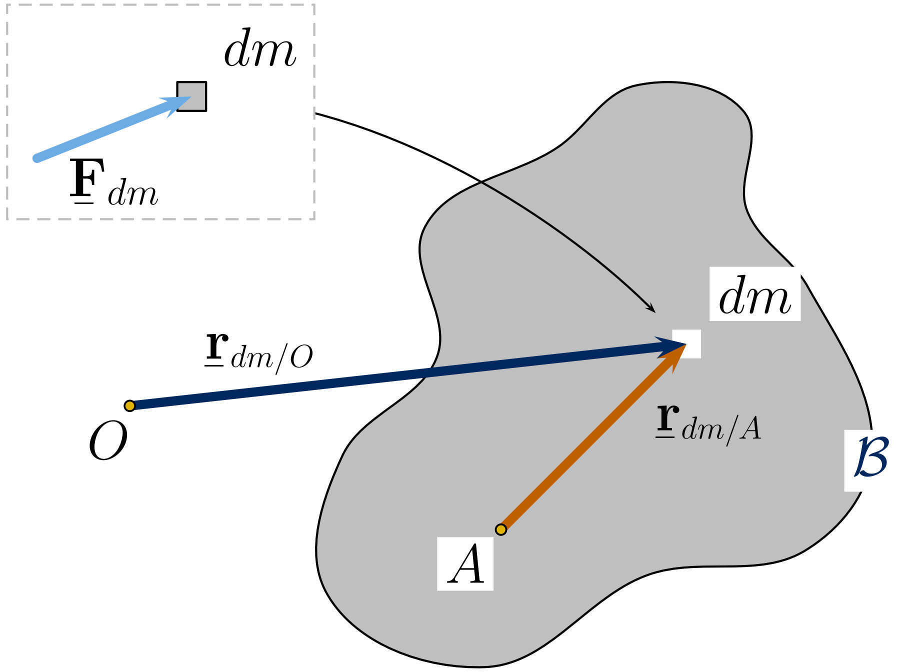
About A is defined as

$$\underline{\mathbf{r}}_{dm/A} \times \{\underline{\mathbf{F}}_{dm} = \underline{\mathbf{a}}_{dm}\, dm\}\,.$$

As before, the total force can be written as

$$\underline{\mathbf{F}}_{dm} = \underline{\mathbf{F}}^{\text{external}}_{dm} + \underline{\mathbf{F}}^{\text{internal}}_{dm},$$

and this can be integrated over the body $\mathcal{B}$



$$\int_{\mathcal{B}} \left\{ \underline{\mathbf{r}}_{dm/A} \times \left\{ \underline{\mathbf{F}}^{\text{external}}_{dm} + \underline{\mathbf{F}}^{\text{internal}}_{dm} \right\} = \underline{\mathbf{r}}_{dm/A} \times \left\{ \left(\underline{\mathbf{a}}_A + \underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A} + \underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \left(\underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A} \right) \right) \right\} \, dm \right\},$$

Here

$$\int_{\mathcal{B}} \left\{ \underline{\mathbf{r}}_{dm/A} \times \underline{\mathbf{F}}^{\text{external}}_{dm} \right\} \equiv \sum \underline{\mathbf{M}}^{\mathcal{B}}_{A}, \qquad \int_{\mathcal{B}} \left\{ \underline{\mathbf{r}}_{dm/A} \times \underline{\mathbf{F}}^{\text{internal}}_{dm} \right\} \equiv \underline{\mathbf{0}},$$

and for planar objects

$$\underline{\mathbf{r}}_{dm/A} \times \left(\underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \left(\underline{\boldsymbol{\omega}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A} \right) \right) = \underline{\mathbf{0}}.$$

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So that Angular Momentum Balance reduces to

$$\sum \underline{\mathbf{M}}_A^{\mathcal{B}} = \int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{dm/A} \times \underline{\mathbf{a}}_A\right)\, dm + \int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{dm/A} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A}\right)\right)\, dm$$

with $\underline{\mathbf{r}}_{dm/A} = \underline{\mathbf{r}}_{G/A} + \underline{\mathbf{r}}_{dm/G}$ (G is the mass center)

$$\sum \underline{\mathbf{M}}_A^{\mathcal{B}} = \underline{\mathbf{r}}_{G/A} \times m\, \underline{\mathbf{a}}_A + \int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{dm/A} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A}\right)\right)\, dm.$$

For *planar* objects

$$\underline{\mathbf{r}}_{dm/A} = s_1\, \hat{\boldsymbol{e}}_1 + s_2\, \hat{\boldsymbol{e}}_2,$$

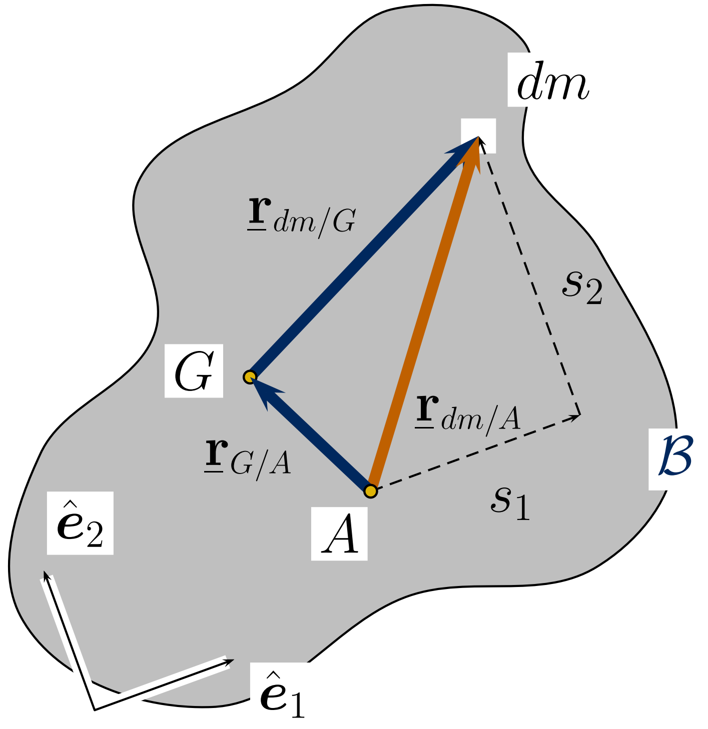
so that, with $\underline{\boldsymbol{\alpha}}_{\mathcal{B}} = \alpha\, \hat{\boldsymbol{k}}$

$$\begin{aligned}\underline{\mathbf{r}}_{dm/A} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A}\right) &= \left(s_1^2 + s_2^2\right)\, \alpha\, \hat{\boldsymbol{k}}, \\ &= \left\|\underline{\mathbf{r}}_{dm/A}\right\|^2 \underline{\boldsymbol{\alpha}}_{\mathcal{B}},\end{aligned}$$

with $\left\|\underline{\mathbf{r}}_{dm/A}\right\| \equiv s$

$$\begin{aligned}\int_{\mathcal{B}} \left(\underline{\mathbf{r}}_{dm/A} \times \left(\underline{\boldsymbol{\alpha}}_{\mathcal{B}} \times \underline{\mathbf{r}}_{dm/A}\right)\right)\, dm &= \left\{\int_{\mathcal{B}} s^2\, dm\right\} \underline{\boldsymbol{\alpha}}_{\mathcal{B}}, \\ &\equiv I_A^{\mathcal{B}}\, \underline{\boldsymbol{\alpha}}_{\mathcal{B}}.\end{aligned}$$

$I_A^{\mathcal{B}}$: Moment of Inertia of $\mathcal{B}$ about A



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Therefore, for planar objects

$$\sum \underline{\mathbf{M}}_A^{\mathcal{B}} = \underline{\mathbf{r}}_{G/A} \times m\, \underline{\mathbf{a}}_A + I_A^{\mathcal{B}}\, \underline{\boldsymbol{\alpha}}_{\mathcal{B}}.$$

The term $\underline{\mathbf{r}}_{G/A} \times m\, \underline{\mathbf{a}}_A$ vanishes in three cases

I. A is the mass center G ($\underline{\mathbf{r}}_{G/A} \equiv \underline{\mathbf{0}}$, $A \equiv G$)

$$\sum \underline{\mathbf{M}}_G^{\mathcal{B}} = I_G^{\mathcal{B}}\, \underline{\boldsymbol{\alpha}}_{\mathcal{B}}.$$

This is *always* valid.

II. A is fixed in the object *and* fixed in the ground ($\underline{\mathbf{a}}_A \equiv \underline{\mathbf{0}}$, $A \equiv O$)

$$\sum \underline{\mathbf{M}}_O^{\mathcal{B}} = I_O^{\mathcal{B}}\, \underline{\boldsymbol{\alpha}}_{\mathcal{B}}.$$

The object is typically pinned at O.

III. If an object is rolling without slip on the ground then with A as the contact point the position $\underline{\mathbf{r}}_{G/A}$ is parallel to the acceleration $\underline{\mathbf{a}}_A$ ($\underline{\mathbf{r}}_{G/A} \parallel \underline{\mathbf{a}}_A$, $A \equiv C$)

$$\sum \underline{\mathbf{M}}_C^{\mathcal{B}} = I_C^{\mathcal{B}}\, \underline{\boldsymbol{\alpha}}_{\mathcal{B}}.$$

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