Projectile Motion

Engineering Mechanics: Dynamics

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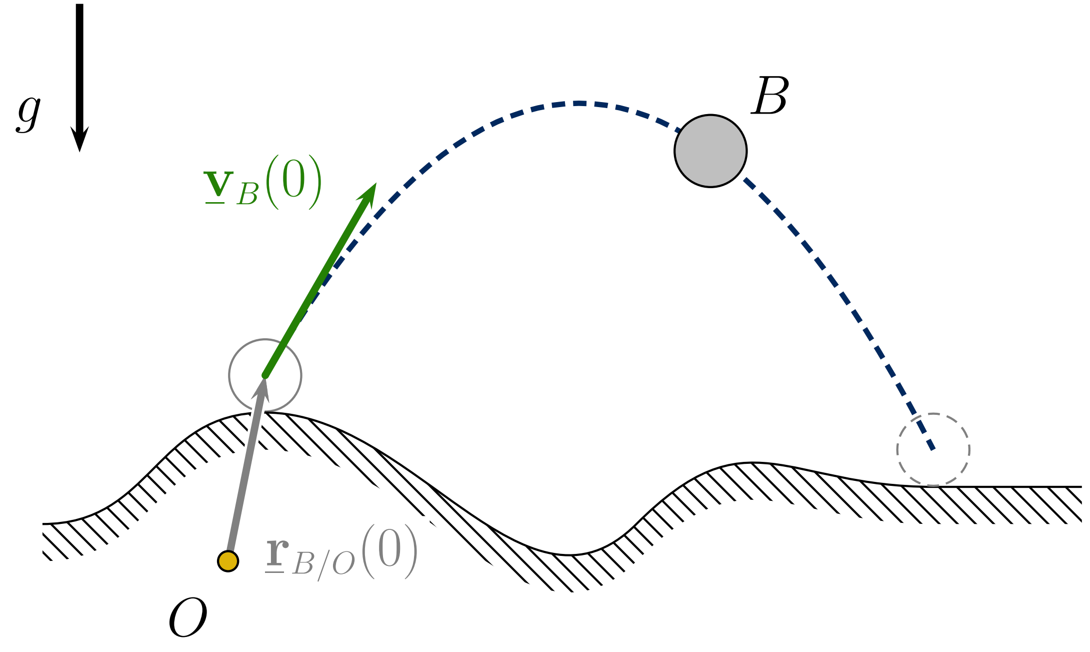
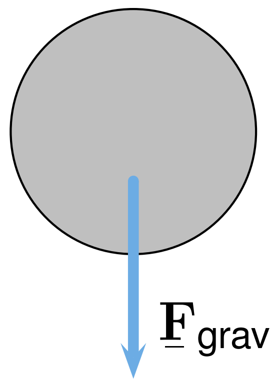
- Describes the motion of a particle subject only to the force due to gravity.
- The trajectory (path through space) depends only on the initial conditions (position and velocity) of the object.

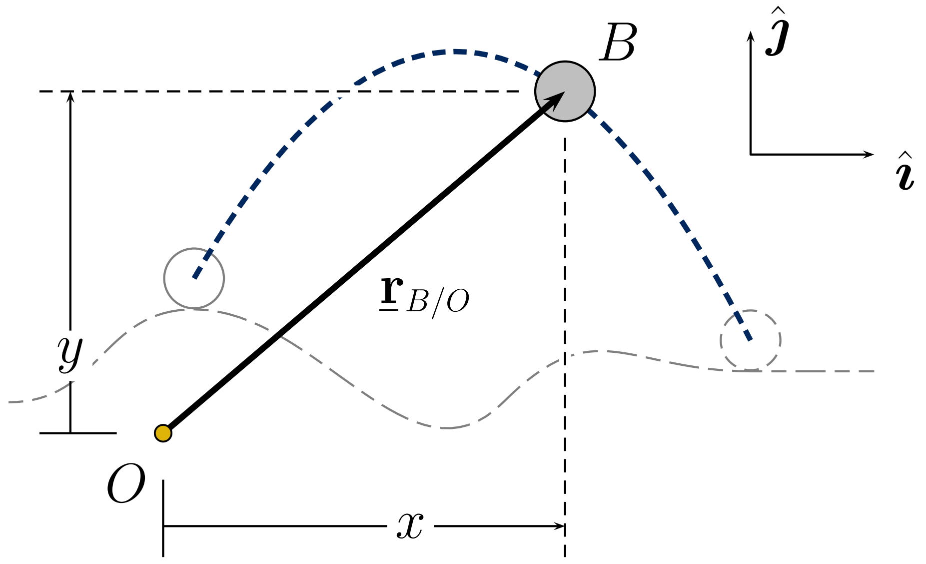


Coordinates and Directions



Only the gravitational force acts on the particle, while the motion of the particle is in the plane, so that we define x and y to describe the displacement of the ball as

$$\underline{\mathbf{r}}_{B/O}(t) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}}.$$



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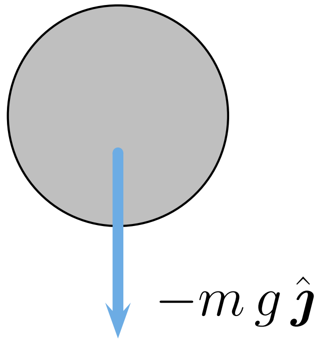
Kinematics/Free Body Diagram

With this, the acceleration of the ball is

$$\underline{\mathbf{a}}_B = \ddot{x}\,\hat{\boldsymbol{\imath}} + \ddot{y}\,\hat{\boldsymbol{\jmath}},$$

and the free body diagram is shown to the right.



Equations of Motion

Finally, applying linear momentum balance to this particle

$$\sum \underline{\mathbf{F}} = -m\,g\,\hat{\boldsymbol{\jmath}} = m\,\left(\ddot{x}\,\hat{\boldsymbol{\imath}} + \ddot{y}\,\hat{\boldsymbol{\jmath}}\right) = m\,\underline{\mathbf{a}}_B,$$

so that

$$\underline{\mathbf{0}} = \Big[m\,\ddot{x}\Big]\,\hat{\boldsymbol{\imath}} + \Big[m\,\ddot{y} + m\,g\Big]\,\hat{\boldsymbol{\jmath}}.$$

Taking components in the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions provides

$\hat{\boldsymbol{\imath}}$ direction $\longrightarrow$ $\ddot{x} = 0,$

$\hat{\boldsymbol{\jmath}}$ direction $\longrightarrow$ $\ddot{y} = -g,$ (constant acceleration),

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Integrating to find the velocity

$$\int_0^t \left\{ \ddot{x}(\tau) = 0 \right\} d\tau \quad \longrightarrow \quad \dot{x}(t) - \dot{x}(0) = 0, \qquad (\dot{x}(0) \equiv v_x),$$

$$\dot{x}(t) = v_x,$$

$$\int_0^t \left\{ \ddot{y}(\tau) = -g \right\} d\tau \quad \longrightarrow \quad \dot{y}(t) - \dot{y}(0) = -g\,t, \qquad (\dot{y}(0) \equiv v_y),$$

$$\dot{y}(t) = -g\,t + v_y,$$

and the position

$$\int_0^t \left\{ \dot{x}(\tau) = v_x \right\} d\tau \quad \longrightarrow \quad x(t) - x(0) = v_x\,t, \qquad (x(0) \equiv x_0)$$

$$x(t) = v_x\,t + x_0,$$

$$\int_0^t \left\{ \dot{y}(\tau) = -g\,\tau + v_y \right\} d\tau \quad \longrightarrow \quad y(t) - y(0) = -\frac{g\,t^2}{2} + v_y\,t, \qquad (y(0) \equiv y_0)$$

$$y(t) = -\frac{g\,t^2}{2} + v_y\,t + y_0,$$

These are valid for any particle undergoing projectile motion

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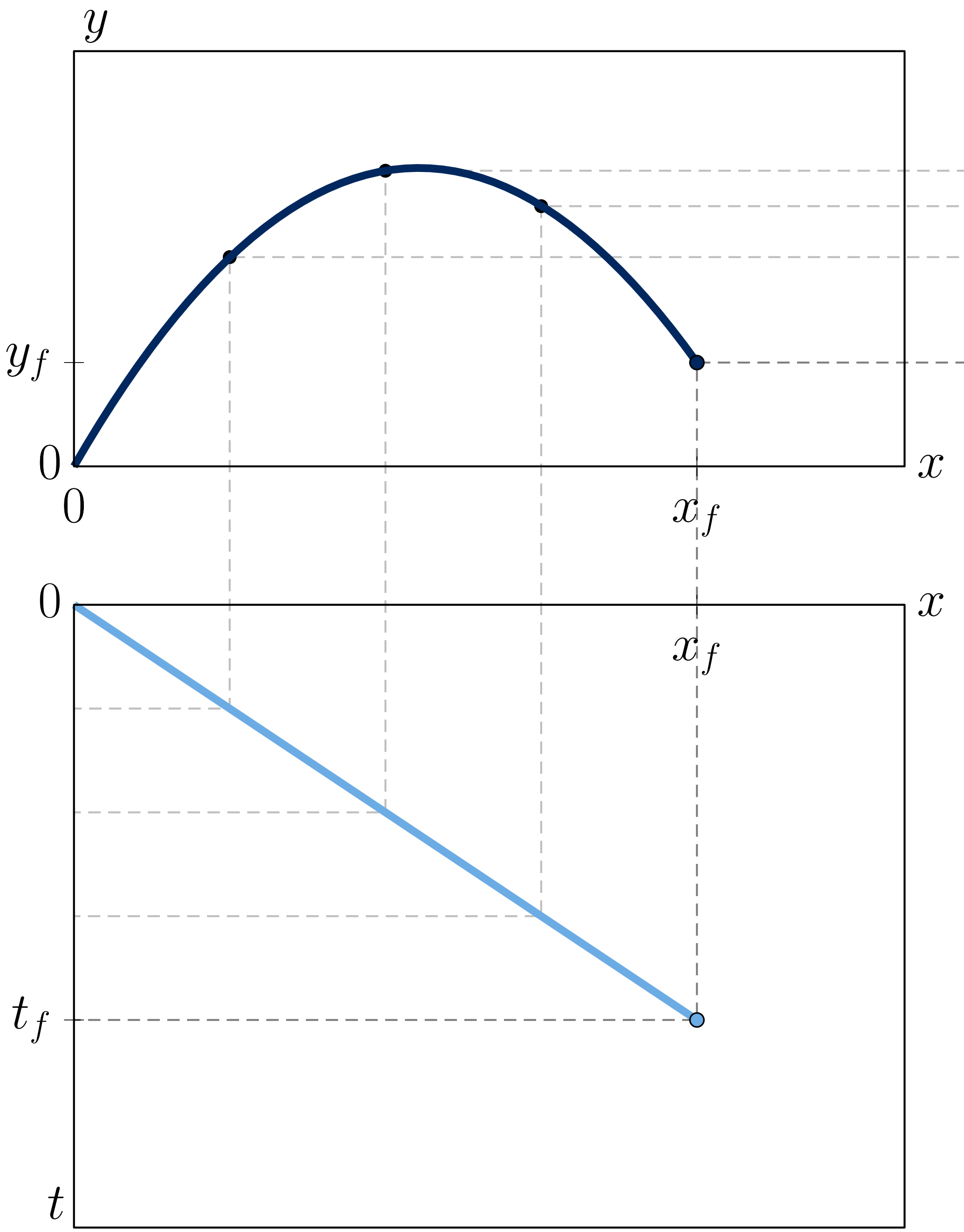

$$(x_0, y_0) = (0, 0)$$

$$x(t) = v_x\, t, \qquad y(t) = -\frac{g\, t^2}{2} + v_y\, t,$$

$$t = \frac{x}{v_x} \quad \longrightarrow \quad y(x) = -\frac{g\, x^2}{2\, v_x^2} + \frac{v_y}{v_x}\, x.$$

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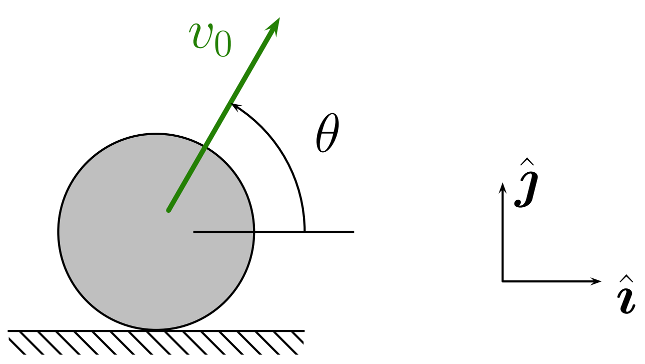
The initial velocity $\underline{\mathbf{v}}_B(0)$ can be represented as

$$\underline{\mathbf{v}}_B(0) = v_x\,\hat{\boldsymbol{\imath}} + v_y\,\hat{\boldsymbol{\jmath}} = v_0\ (\mathrm{C}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{S}_\theta\ \hat{\boldsymbol{\jmath}})\,,$$



where v_0 is the initial speed and θ is the initial angle of inclination. Therefore

$$\dot{x}(0) \equiv v_x = v_0\ \mathrm{C}_\theta, \qquad \dot{y}(0) \equiv v_y = v_0\ \mathrm{S}_\theta\,.$$

In terms of the initial speed and inclination, with $(x_0, y_0) = (0, 0)$

$$\dot{x}(t) = v_0\ \mathrm{C}_\theta, \qquad \dot{y}(t) = -g\,t + v_0\ \mathrm{S}_\theta,$$

$$x(t) = v_0\ \mathrm{C}_\theta\ t, \qquad y(t) = -\frac{g\,t^2}{2} + v_0\ \mathrm{S}_\theta\ t,$$

while the trajectory becomes

$$y(x) = -\frac{g\,x^2}{2\,v_x^2} + \frac{v_y}{v_x}\,x = -\frac{g\,x^2}{2\,v_0^2\ \mathrm{C}_\theta^2} + \mathrm{T}_\theta\ x.$$

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The motion can also be defined in terms of auxiliary conditions, rather than the initial conditions.

- Auxiliary conditions provide equations that can be used to solve for the inital conditions

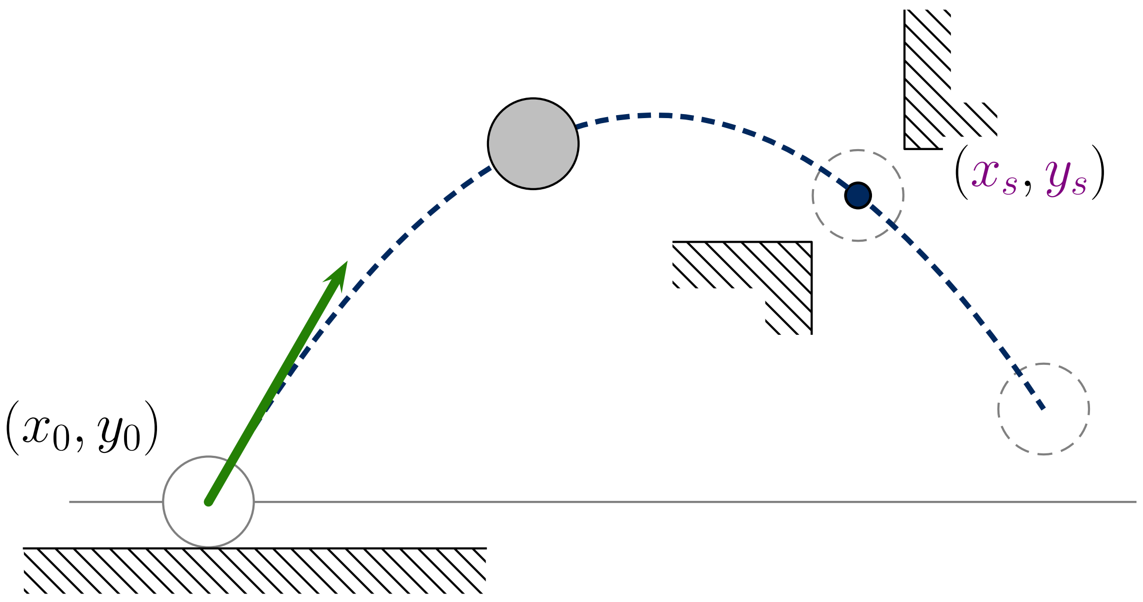
Secondary Position

If the trajectory passes through a secondary point (x_s, y_s) at some time t_s, then

$$x(t_s) = x_s = v_x\, t_s + x_0,$$

$$y(t_s) = y_s = -\frac{g\, {t_s}^2}{2} + v_y\, t_s + y_0,$$

$$\longrightarrow \quad y_s = -\frac{g\,(x_s - x_0)^2}{2\,v_x^2} + \frac{v_y}{v_x}\,(x_s - x_0) + y_0.$$



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Maximum Height

If the maximum height $y = y_\star$ occurs at time $t = t_\star$ (with $x_0 = 0$, and $y_0 = 0$)

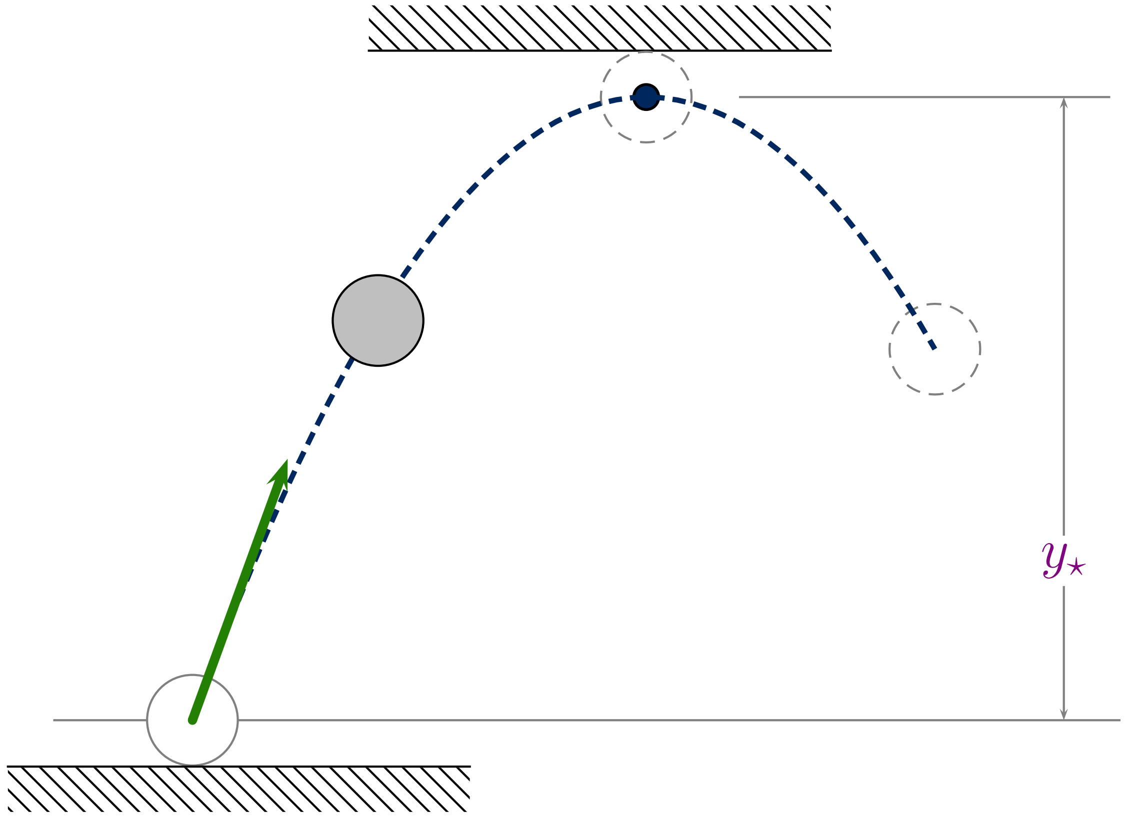
$$\dot{y}(t_\star) = 0 = -g\, t_\star + v_y, \quad \longrightarrow \quad t_\star = \frac{v_y}{g},$$

so that

$$y_\star = y(t_\star) = \frac{v_y^2}{2\,g} = \frac{v_0^2\, \mathrm{C}_\theta^2}{2\,g},$$

and

$$x(t_\star) = v_x\, t_\star = \frac{v_x\, v_y}{g} = \frac{v_0^2\, \mathrm{S}_\theta\, \mathrm{C}_\theta}{g}.$$



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