# Vector Derivatives

## Engineering Mechanics: Dynamics

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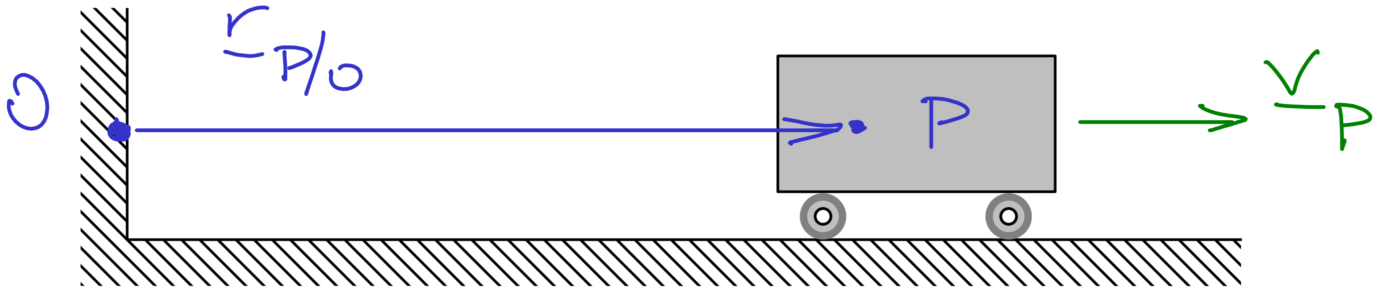
# Vectors can change in both *magnitude* and *direction*

$$\underline{\text{VECTOR}} = (\text{MAGNITUDE}) \cdot \underline{\text{DIRECTION}}$$

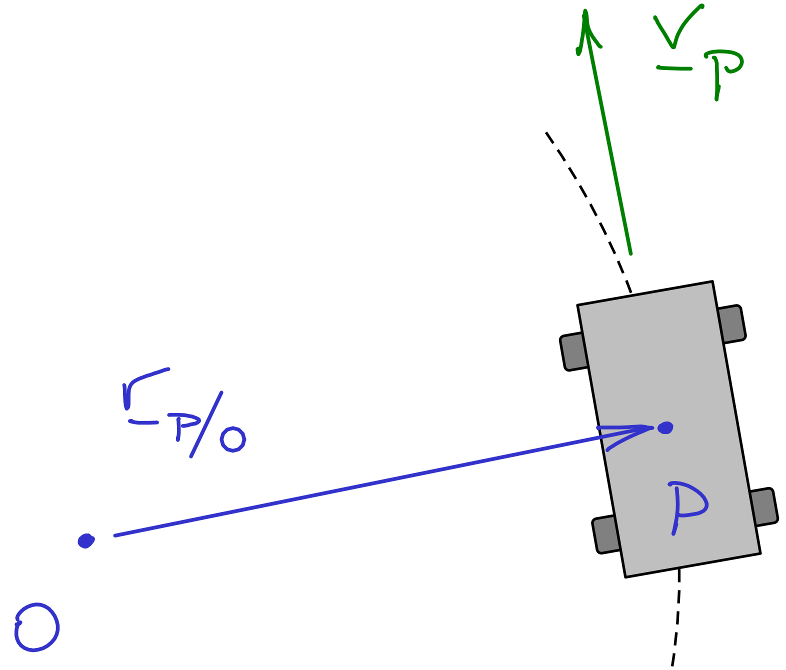
$$\frac{d}{dt}\left(\underline{\text{VECTOR}}\right) = \frac{d}{dt}\left(\text{MAGNITUDE}\right) \cdot \underline{\text{DIRECTION}} + \text{MAGNITUDE} \cdot \frac{d}{dt}\left(\underline{\text{DIRECTION}}\right)$$

## Straight-line motion




$\underline{r}_{P/O}$ CHANGES MAGNITUDE BUT NOT DIRECTION

## Circular motion




$\underline{r}_{P/O}$ CHANGES DIRECTION BUT NOT MAGNITUDE

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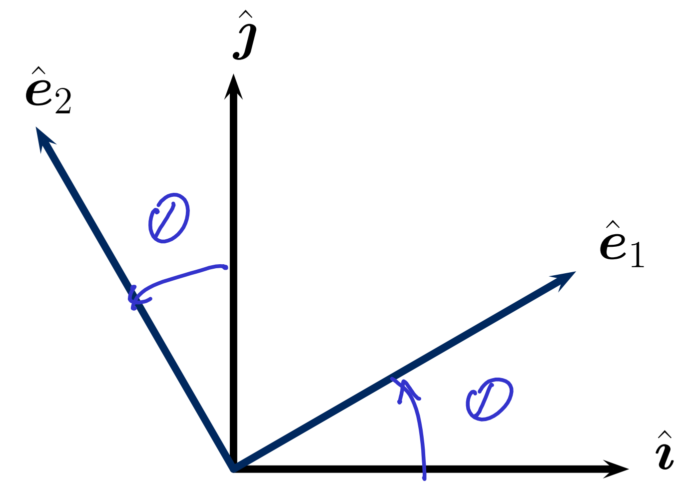
# Change in Direction

LET $(\hat{e}_1, \hat{e}_2)$ BE ORIENTED BY AN ANGLE $\theta$ WITH RESPECT TO $(\hat{\imath}, \hat{\jmath})$




$$\hat{e}_1 = C_\theta \hat{\imath} + S_\theta \hat{\jmath}$$

$$\hat{e}_2 = -S_\theta \hat{\imath} + C_\theta \hat{\jmath}$$

$(\hat{e}_1, \hat{e}_2)$ VARY AS $\theta$ VARIES

$$\frac{d}{dt}(\hat{e}_1) = \frac{d}{dt}\left(C_\theta \hat{\imath} + S_\theta \hat{\jmath}\right)$$

$$= \frac{d}{dt}(C_\theta)\hat{\imath} + \cancel{C_\theta \frac{d}{dt}(\hat{\imath})}^{\,0} + \frac{d}{dt}(S_\theta)\hat{\jmath} + \cancel{S_\theta \frac{d}{dt}(\hat{\jmath})}^{\,0}$$

$\frac{d\hat{\imath}}{dt} = 0, \quad \frac{d\hat{\jmath}}{dt} = 0$

$$= -S_\theta \frac{d\theta}{dt}\hat{\imath} + C_\theta \frac{d\theta}{dt}\hat{\jmath} = \dot{\theta}\left(-S_\theta \hat{\imath} + C_\theta \hat{\jmath}\right) \equiv \dot{\theta}\hat{e}_2$$

$\dot{(\;)} \equiv \frac{d}{dt}(\;)$

LIKEWISE $\quad \frac{d}{dt}(\hat{e}_2) = -\dot{\theta}\hat{e}_1$

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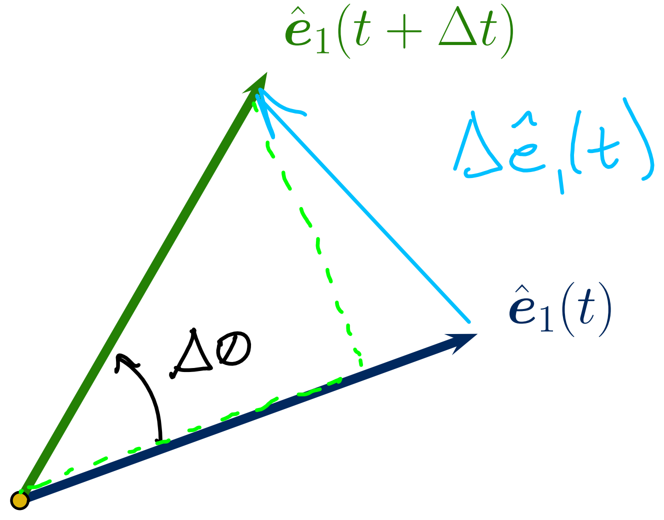
# Reconsider using the definition of a derivative

LET $\Delta\theta$ BE THE ANGLE TRAVERSED BY $\hat{e}_1$ DURING A TIME INTERVAL $\Delta t$




$$\hat{e}_1(t+\Delta t) = C_{\Delta\theta}\,\hat{e}_1(t) + S_{\Delta\theta}\,\hat{e}_2(t)$$

$$\Delta\hat{e}_1(t) = \hat{e}_1(t+\Delta t) - \hat{e}_1(t) = (C_{\Delta\theta}-1)\,\hat{e}_1(t) + S_{\Delta\theta}\,\hat{e}_2(t)$$

THEREFORE

$$\frac{d}{dt}(\hat{e}_1) = \lim_{\Delta t\to 0}\frac{\Delta\hat{e}_1(t)}{\Delta t} = \lim_{\Delta t\to 0}\left\{\frac{C_{\Delta\theta}-1}{\Delta t}\,\hat{e}_1(t) + \frac{S_{\Delta\theta}}{\Delta t}\,\hat{e}_2(t)\right\}$$

$$C_{\Delta\theta} \sim 1 - \frac{\Delta\theta^2}{2} + \cdots$$

$$S_{\Delta\theta} \sim \Delta\theta$$

$$= \lim_{\Delta t\to 0}\left\{\frac{\Delta\theta}{\Delta t}\,\hat{e}_2(t)\right\} \equiv \dot{\theta}\,\hat{e}_2(t)$$

IDENTICAL TO THE PREVIOUS RESULT

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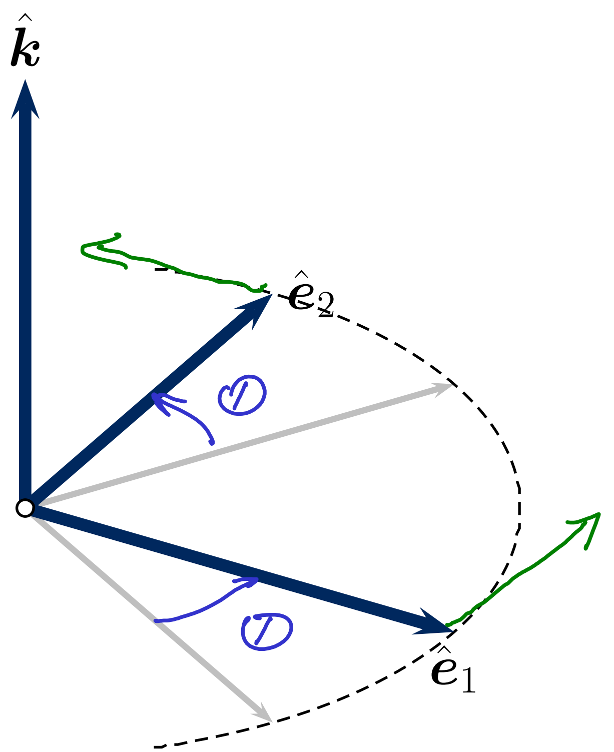
THE CHANGE IN A DIRECTION IS ALWAYS PERPENDICULAR TO THE DIRECTION ITSELF

DEFINE THE ANGULAR VELOCITY OF $\hat{e}_1$ AS

$$\underline{\omega} = \dot{\theta}\hat{k}$$




$\dot{\theta}$ = ANGULAR SPEED

$\hat{k}$ = SPIN AXIS; PERPENDICULAR TO THE (INSTANTANEOUS) PLANE OF MOTION FOR $(\hat{e}_1, \hat{e}_2)$

$$\frac{d}{dt}(\hat{e}_1) = \dot{\theta}\hat{e}_2 = (\dot{\theta}\hat{k}) \times \hat{e}_1 = \underline{\omega} \times \hat{e}_1$$

$$\frac{d}{dt}(\hat{e}_2) = -\dot{\theta}\hat{e}_1 = (\dot{\theta}\hat{k}) \times \hat{e}_2 = \underline{\omega} \times \hat{e}_2$$

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LET $\underline{s} = s_1\hat{e}_1 + s_2\hat{e}_2$ & ASSUME $\underline{\omega}_B$ IS KNOWN

$$
{}^{G}\frac{d}{dt}(\underline{s}) = {}^{G}\frac{d}{dt}\left(s_1\hat{e}_1 + s_2\hat{e}_2\right)
$$

$$
= {}^{G}\frac{d}{dt}(s_1)\,\hat{e}_1 + s_1\,{}^{G}\frac{d}{dt}(\hat{e}_1) + {}^{G}\frac{d}{dt}(s_2)\,\hat{e}_2 + s_2\,{}^{G}\frac{d}{dt}(\hat{e}_2)
$$

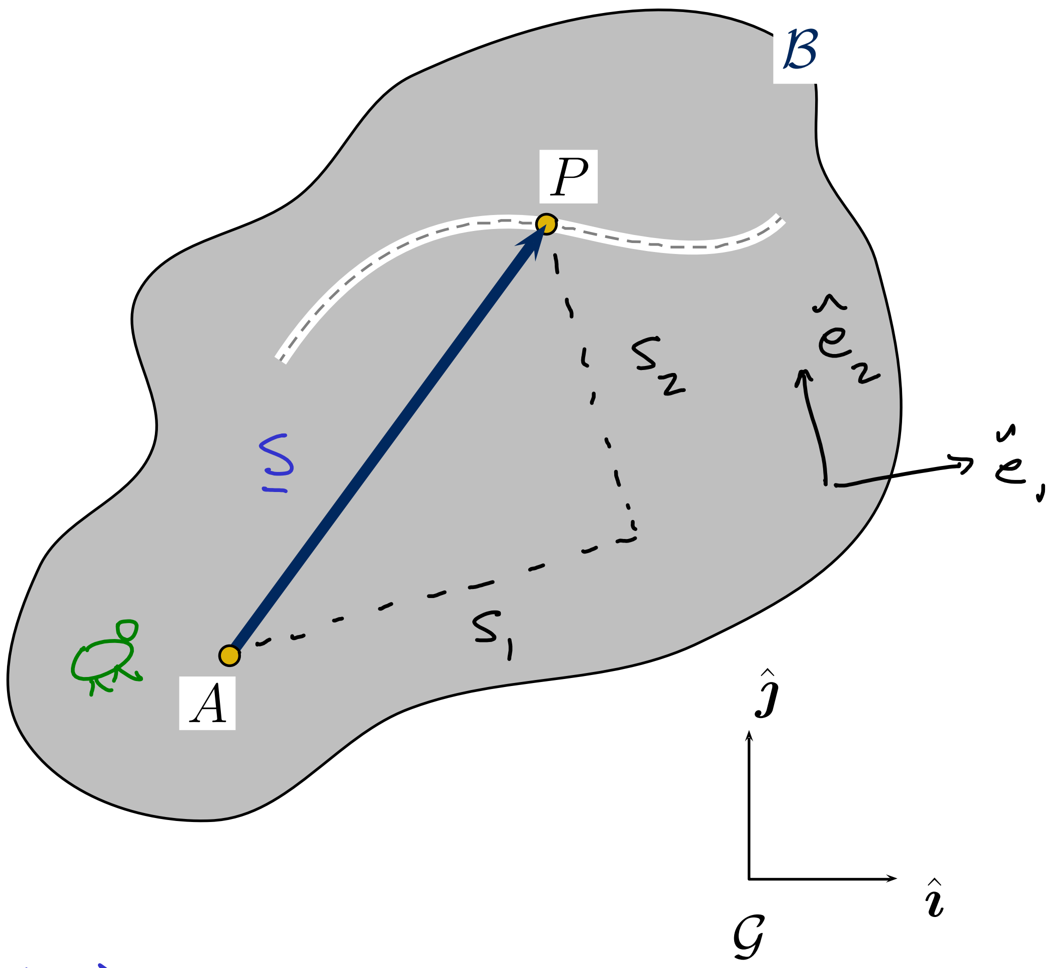
$$
= \left\{ {}^{G}\frac{d}{dt}(s_1)\,\hat{e}_1 + {}^{G}\frac{d}{dt}(s_2)\,\hat{e}_2 \right\} + \left\{ s_1\,{}^{G}\frac{d}{dt}(\hat{e}_1) + s_2\,{}^{G}\frac{d}{dt}(\hat{e}_2) \right\}
$$

$$
= \left\{ \frac{d}{dt}(s_1)\,\hat{e}_1 + \frac{d}{dt}(s_2)\,\hat{e}_2 \right\}
$$

DERIVATIVE WITH RESPECT TO B

$$
+ \left\{ s_1\left(\underline{\omega}_B \times \hat{e}_1\right) + s_2\left(\underline{\omega}_B \times \hat{e}_2\right) \right\} = {}^{B}\frac{d}{dt}(\underline{s}) + \underline{\omega}_B \times \underline{s}
$$

$$
\underline{\omega}_B \times \left(s_1\hat{e}_1 + s_2\hat{e}_2\right)
$$

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