Vector Derivatives

Engineering Mechanics: Dynamics

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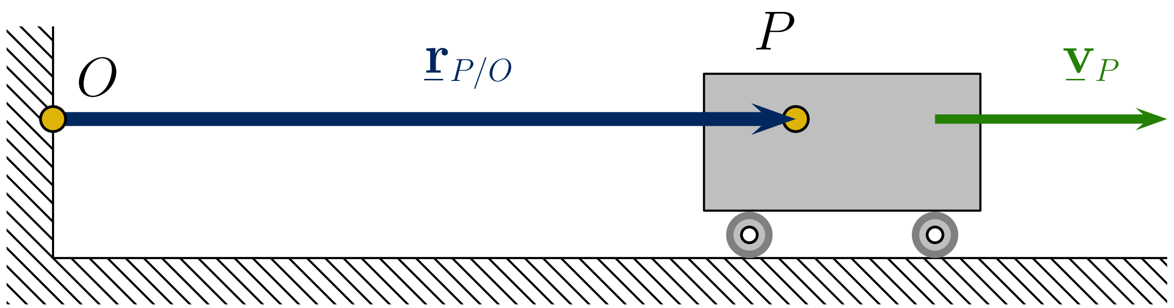
Vectors can change in both *magnitude* and *direction*

$$\underline{\textbf{Vector}} = (\text{Magnitude}) \cdot (\underline{\textbf{Direction}})$$

$$\longrightarrow \quad \frac{d}{dt}(\underline{\textbf{Vector}}) = \left\{ \frac{d}{dt}(\text{Magnitude}) \cdot (\underline{\textbf{Direction}}) \right\} + \left\{ (\text{Magnitude}) \cdot \frac{d}{dt}(\underline{\textbf{Direction}}) \right\}.$$

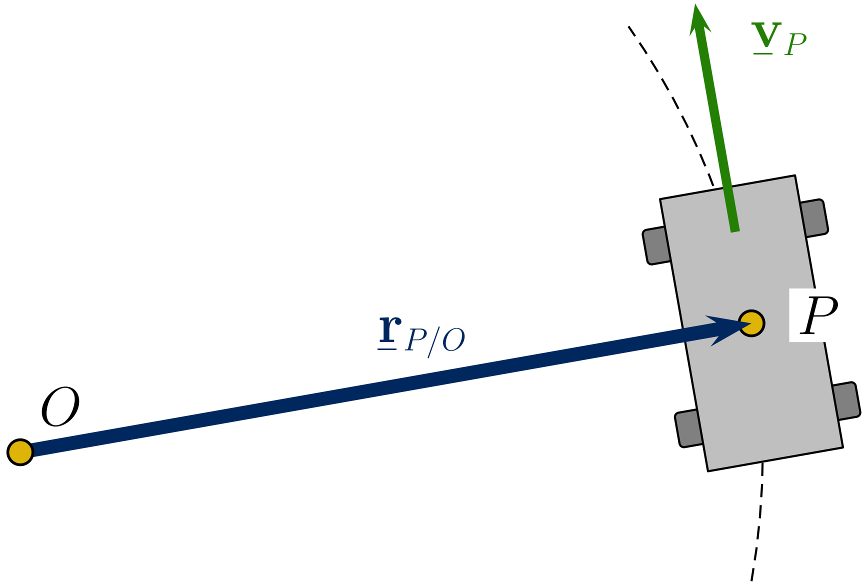
Straight-line motion

$\underline{\mathbf{r}}_{P/O}$ changes magnitude but not direction



Circular motion

$\underline{\mathbf{r}}_{P/O}$ changes direction but not magnitude



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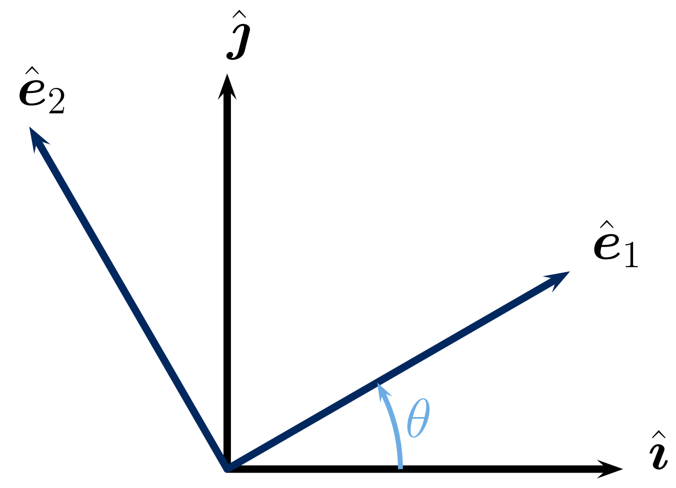
Change in Direction

Consider directions $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$ at angle θ with respect to $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$, so that

$$\hat{\boldsymbol{e}}_1 = \mathrm{C}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{S}_\theta\ \hat{\boldsymbol{\jmath}},$$
$$\hat{\boldsymbol{e}}_2 = -\,\mathrm{S}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{C}_\theta\ \hat{\boldsymbol{\jmath}},$$

$(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$ are fixed, but $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$ can change as θ varies.



Therefore

$$\begin{aligned}
\frac{d\,\hat{\boldsymbol{e}}_1}{dt} &= \frac{d}{dt}\left\{\mathrm{C}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{S}_\theta\ \hat{\boldsymbol{\jmath}}\right\},\\
&= \frac{d(\mathrm{C}_\theta)}{dt}\,\hat{\boldsymbol{\imath}} + \mathrm{C}_\theta\,\frac{d\hat{\boldsymbol{\imath}}}{dt} + \frac{d(\mathrm{S}_\theta)}{dt}\,\hat{\boldsymbol{\jmath}} + \mathrm{S}_\theta\,\frac{d\hat{\boldsymbol{\jmath}}}{dt}, \qquad \left[\frac{d\hat{\boldsymbol{\imath}}}{dt} = \frac{d\hat{\boldsymbol{\jmath}}}{dt} \equiv \mathbf{0}\right]\\
&= -\frac{d\theta}{dt}\,\mathrm{S}_\theta\ \hat{\boldsymbol{\imath}} + \frac{d\theta}{dt}\,\mathrm{C}_\theta\ \hat{\boldsymbol{\jmath}} = \dot{\theta}\,\left(-\,\mathrm{S}_\theta\ \hat{\boldsymbol{\imath}} + \mathrm{C}_\theta\ \hat{\boldsymbol{\jmath}}\right) \equiv \dot{\theta}\,\hat{\boldsymbol{e}}_2 \qquad \frac{d(\ )}{dt} \equiv \dot{(\ )}
\end{aligned}$$

Likewise

$$\frac{d\hat{\boldsymbol{e}}_2}{dt} = -\dot{\theta}\,\hat{\boldsymbol{e}}_1.$$

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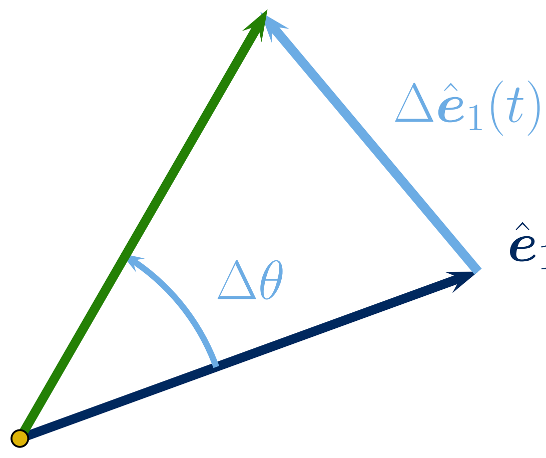
Reconsider using the definition of a derivative

Let $\Delta\theta$ be an angle traversed by $\hat{\boldsymbol{e}}_1$ during a (small) time interval Δt, so that

$$\hat{\boldsymbol{e}}_1(t+\Delta t) = \mathrm{C}_{\Delta\theta}\ \hat{\boldsymbol{e}}_1(t) + \mathrm{S}_{\Delta\theta}\ \hat{\boldsymbol{e}}_2(t),$$

$$\longrightarrow \qquad \Delta\hat{\boldsymbol{e}}_1(t) = \hat{\boldsymbol{e}}_1(t+\Delta t) - \hat{\boldsymbol{e}}_1(t),$$
$$= (\mathrm{C}_{\Delta\theta} - 1)\ \hat{\boldsymbol{e}}_1(t) + (\mathrm{S}_{\Delta\theta})\ \hat{\boldsymbol{e}}_2(t).$$



Therefore

$$\frac{d\hat{\boldsymbol{e}}_1}{dt} = \lim_{\Delta t \to 0}\left(\frac{\hat{\boldsymbol{e}}_1(t+\Delta t) - \hat{\boldsymbol{e}}_1(t)}{\Delta t}\right) = \lim_{\Delta t\to 0}\left\{\frac{(\mathrm{C}_{\Delta\theta} - 1)}{\Delta t}\,\hat{\boldsymbol{e}}_1(t) + \frac{\mathrm{S}_{\Delta\theta}}{\Delta t}\,\hat{\boldsymbol{e}}_2(t)\right\},$$

As $\Delta\theta \to 0$, $\mathrm{C}_{\Delta\theta} \to 1 - (\Delta\theta)^2/2 + \cdots$ and $\mathrm{S}_{\Delta\theta} \to \Delta\theta - (\Delta\theta)^3/6 + \cdots$

$$= \lim_{\Delta t\to 0}\left\{\frac{\Delta\theta}{\Delta t}\right\}\ \hat{\boldsymbol{e}}_2(t) \equiv \dot{\theta}\,\hat{\boldsymbol{e}}_2.$$

Identical to the previous geometric construction

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The change in a direction is *always* perpendicular to the direction itself

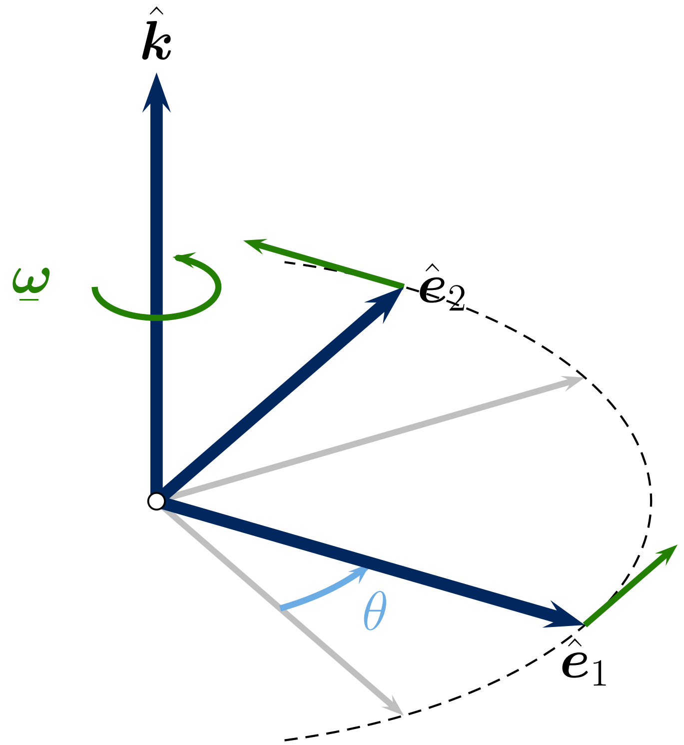
Define the *angular velocity* of $\hat{e}_1$ as $\underline{\boldsymbol{\omega}} = \dot{\theta}\,\hat{\boldsymbol{k}}$, with

- $\dot{\theta}$: angular speed
- $\hat{\boldsymbol{k}}$: spin axis; perpendicular to the (instantaneous) plane of motion for $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$

then

$$\frac{d\hat{\boldsymbol{e}}_1}{dt} \equiv \underline{\boldsymbol{\omega}} \times \hat{\boldsymbol{e}}_1 = \left(\dot{\theta}\,\hat{\boldsymbol{k}}\right) \times \hat{\boldsymbol{e}}_1 = \dot{\theta}\,\hat{\boldsymbol{e}}_2,$$

$$\frac{d\hat{\boldsymbol{e}}_2}{dt} \equiv \underline{\boldsymbol{\omega}} \times \hat{\boldsymbol{e}}_2 = \left(\dot{\theta}\,\hat{\boldsymbol{k}}\right) \times \hat{\boldsymbol{e}}_2 = -\dot{\theta}\,\hat{\boldsymbol{e}}_1$$



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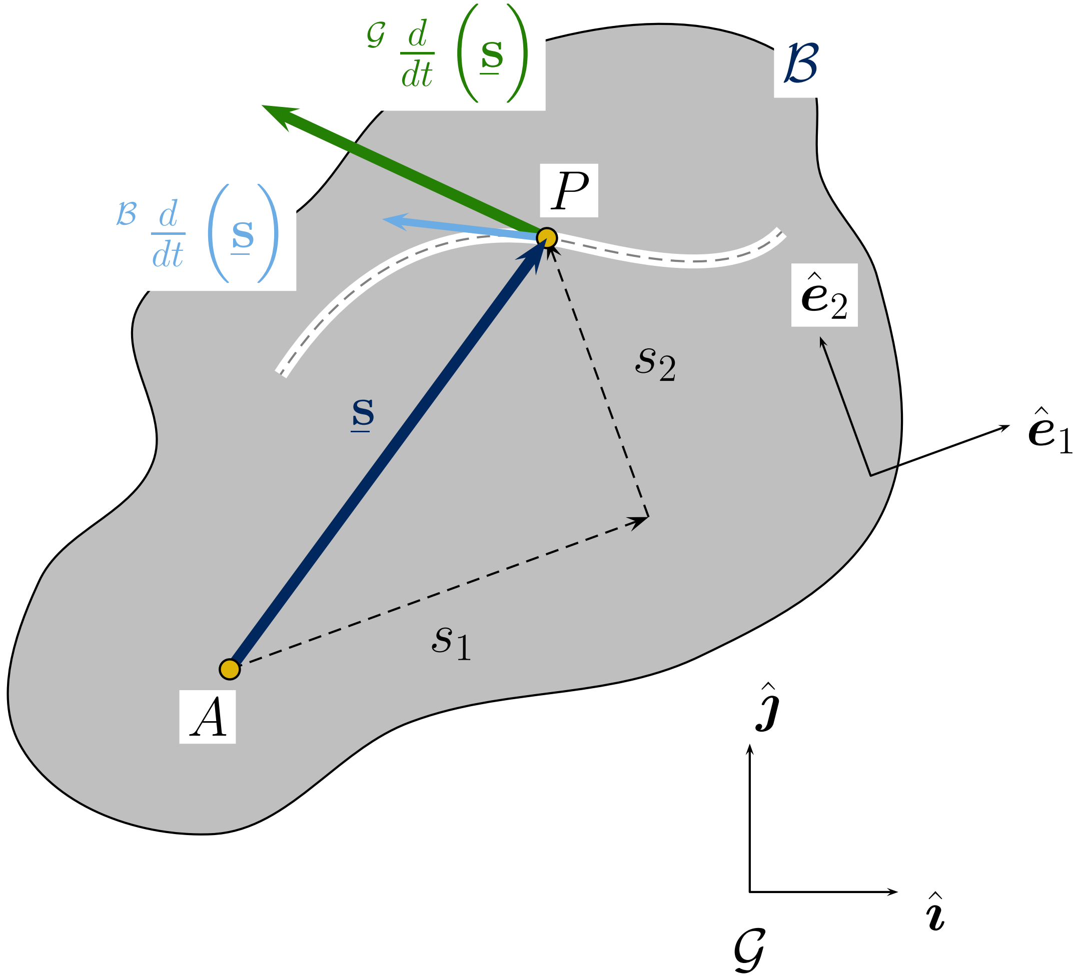
Let a vector $\underline{\mathbf{s}}$ be written in terms of directions $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$ fixed in a (moving) frame of reference $\mathcal{B}$,
$\underline{\mathbf{s}} = s_1\,\hat{\boldsymbol{e}}_1 + s_2\,\hat{\boldsymbol{e}}_2$.

$$
\begin{aligned}
{}^{\mathcal{G}}\frac{d}{dt}\left(\underline{\mathbf{s}}\right) &= {}^{\mathcal{G}}\frac{d}{dt}\left(s_1\,\hat{\boldsymbol{e}}_1 + s_2\,\hat{\boldsymbol{e}}_2\right), \\
&= {}^{\mathcal{G}}\frac{d}{dt}\left(s_1\right)\,\hat{\boldsymbol{e}}_1 + s_1\,{}^{\mathcal{G}}\frac{d}{dt}\left(\hat{\boldsymbol{e}}_1\right) \\
&\quad + {}^{\mathcal{G}}\frac{d}{dt}\left(s_2\right)\,\hat{\boldsymbol{e}}_2 + s_2\,{}^{\mathcal{G}}\frac{d}{dt}\left(\hat{\boldsymbol{e}}_2\right), \\
&= \left\{{}^{\mathcal{G}}\frac{d}{dt}\left(s_1\right)\,\hat{\boldsymbol{e}}_1 + {}^{\mathcal{G}}\frac{d}{dt}\left(s_2\right)\,\hat{\boldsymbol{e}}_2\right\} \\
&\quad + \left\{s_1\,{}^{\mathcal{G}}\frac{d}{dt}\left(\hat{\boldsymbol{e}}_1\right) + s_2\,{}^{\mathcal{G}}\frac{d}{dt}\left(\hat{\boldsymbol{e}}_2\right)\right\}, \\
&= \underbrace{\left\{\frac{d}{dt}\left(s_1\right)\,\hat{\boldsymbol{e}}_1 + \frac{d}{dt}\left(s_2\right)\,\hat{\boldsymbol{e}}_2\right\}}_{\text{derivative with respect to } \mathcal{B}} \\
&\quad + \underbrace{\left\{s_1\left(\underline{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{G}} \times \hat{\boldsymbol{e}}_1\right) + s_2\left(\underline{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{G}} \times \hat{\boldsymbol{e}}_2\right)\right\}}_{\underline{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{G}} \times (s_1\,\hat{\boldsymbol{e}}_1 + s_2\,\hat{\boldsymbol{e}}_2)} \equiv {}^{\mathcal{B}}\frac{d}{dt}\left(\underline{\mathbf{s}}\right) + \underline{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{G}} \times \underline{\mathbf{s}}.
\end{aligned}
$$



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