Vectors

Engineering Mechanics: Dynamics

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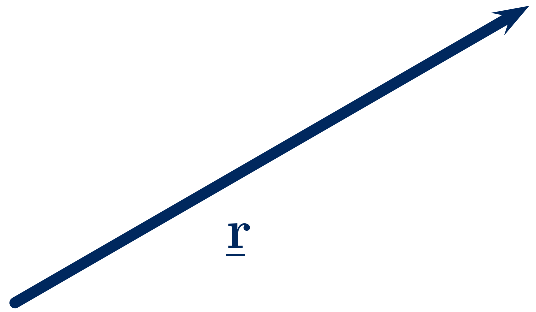
The elements of dynamics (e.g. position, acceleration, forces, moments) are described by *vectors*.

Vector: a mathematical quantity that describes multiple independent variables

- used to describe physical objects with magnitude and direction
- obey laws of vector addition and scalar multiplication
- follow additional operations including dot and cross products

The vector $\underline{\mathbf{r}}$ can be defined

- geometrically in space (i.e., an arrow)
- independently of how *we* choose to measure it



Vectors are denoted with an underbar, like $\underline{\mathbf{r}}$, except for unit vectors (length 1), which are denoted with an overhat, as $\hat{\boldsymbol{\imath}}$.

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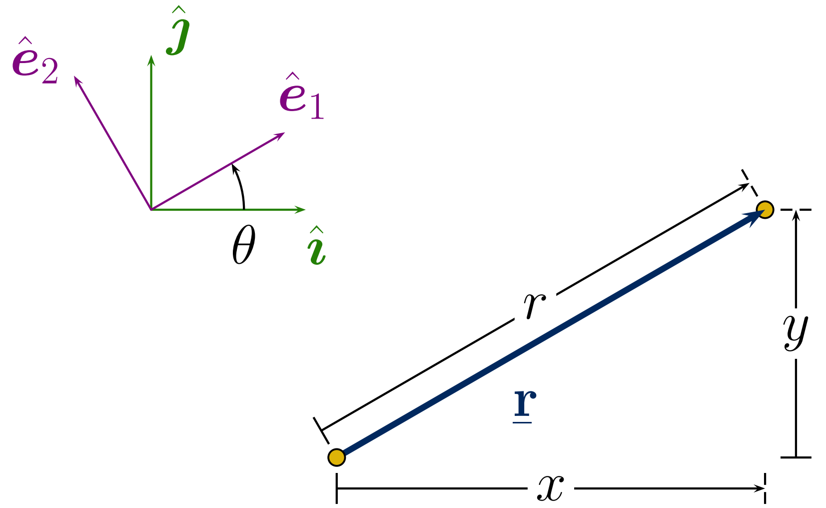
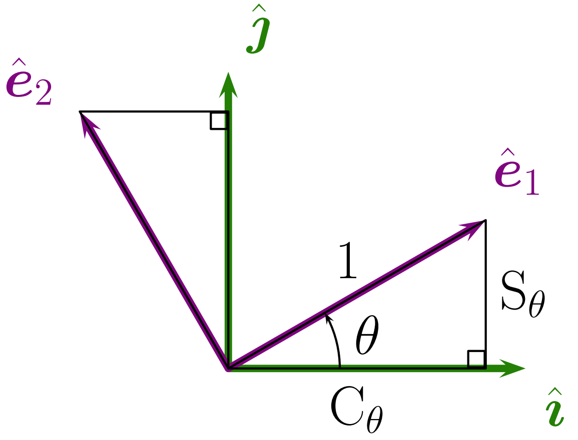
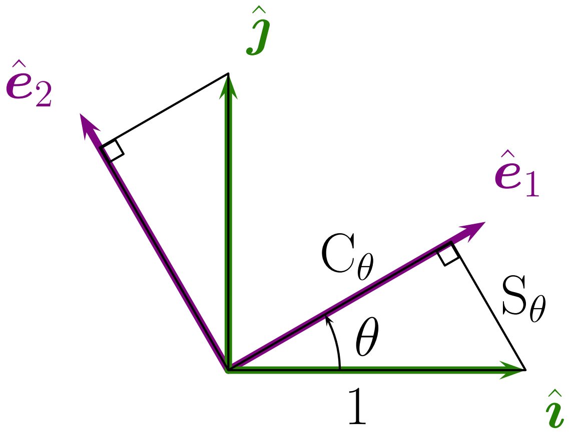
Vectors are specified with respect to known *basis directions*.

- $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$: Cartesian directions (fixed in space)
- $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$: Polar directions (move with $\underline{\mathbf{r}}$)



$$\underline{\mathbf{r}} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} = r\,\hat{\boldsymbol{e}}_1$$





These directions can be related as $(\cos\theta \equiv C_\theta, \sin\theta \equiv S_\theta)$

$$\hat{\boldsymbol{e}}_1 = C_\theta\ \hat{\boldsymbol{\imath}} + S_\theta\ \hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\imath}} = C_\theta\ \hat{\boldsymbol{e}}_1 - S_\theta\ \hat{\boldsymbol{e}}_2,$$
$$\hat{\boldsymbol{e}}_2 = -\,S_\theta\ \hat{\boldsymbol{\imath}} + C_\theta\ \hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\jmath}} = S_\theta\ \hat{\boldsymbol{e}}_1 + C_\theta\ \hat{\boldsymbol{e}}_2,$$

so that

$$\begin{aligned}\underline{\mathbf{r}} &= x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} = r\,\hat{\boldsymbol{e}}_1,\\ &= r\,\left(C_\theta\ \hat{\boldsymbol{\imath}} + S_\theta\ \hat{\boldsymbol{\jmath}}\right), \quad \longrightarrow \quad x = r\ C_\theta, \quad y = r\ S_\theta\,.\end{aligned}$$

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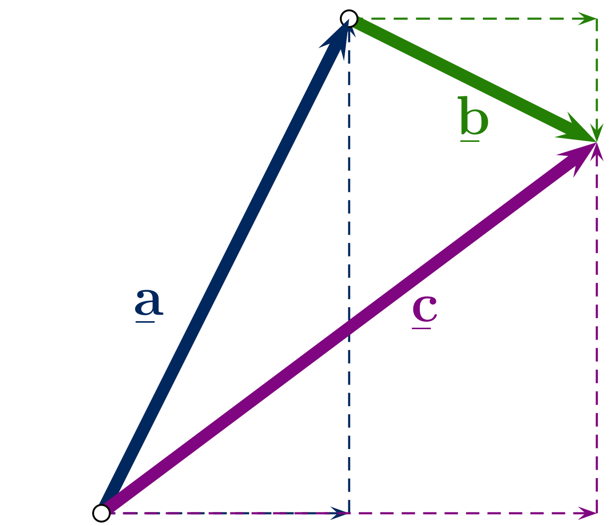
Vector Addition



$$\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$$

These are defined independent of any directions or coordinates, but with

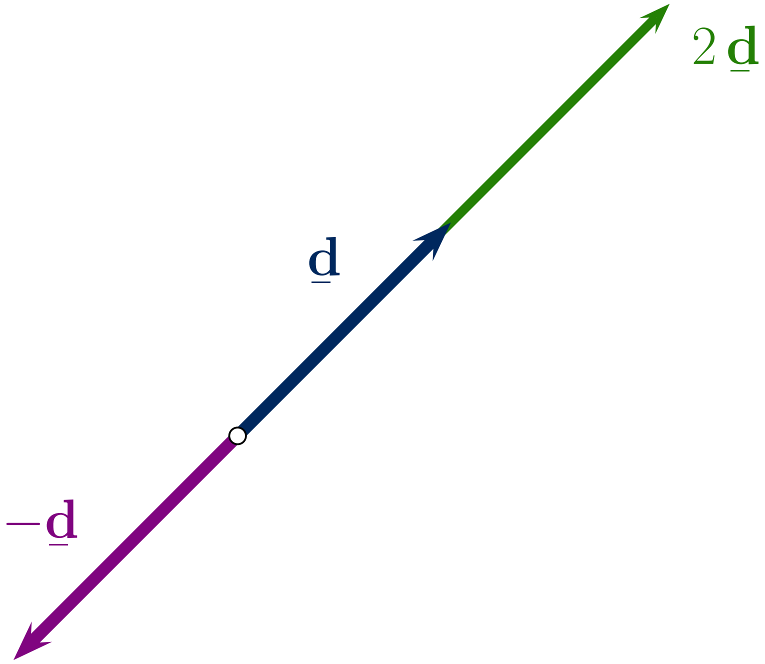
$$\underline{\mathbf{a}} = x_A\,\hat{\boldsymbol{\imath}} + y_A\,\hat{\boldsymbol{\jmath}}, \quad \underline{\mathbf{b}} = x_B\,\hat{\boldsymbol{\imath}} + y_B\,\hat{\boldsymbol{\jmath}}, \quad \underline{\mathbf{c}} = x_C\,\hat{\boldsymbol{\imath}} + y_C\,\hat{\boldsymbol{\jmath}},$$

then

$$x_C\,\hat{\boldsymbol{\imath}} + y_C\,\hat{\boldsymbol{\jmath}} = \left(x_A\,\hat{\boldsymbol{\imath}} + y_A\,\hat{\boldsymbol{\jmath}}\right) + \left(x_B\,\hat{\boldsymbol{\imath}} + y_B\,\hat{\boldsymbol{\jmath}}\right),$$

$$\longrightarrow \quad x_C = x_A + x_B, \quad y_C = y_A + y_B$$

Scalar multiplication



$$\underline{\mathbf{d}} = x_D\,\hat{\boldsymbol{\imath}} + y_D\,\hat{\boldsymbol{\jmath}}, \quad \longrightarrow \quad \begin{aligned} 2\,\underline{\mathbf{d}} &= (2\,x_D)\,\hat{\boldsymbol{\imath}} + (2\,y_D)\,\hat{\boldsymbol{\jmath}}, \\ -\underline{\mathbf{d}} &= (-x_D)\,\hat{\boldsymbol{\imath}} + (-y_D)\,\hat{\boldsymbol{\jmath}}. \end{aligned}$$

Leaves the direction unchanged

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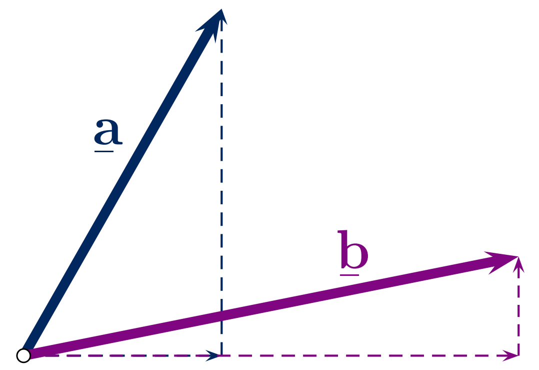
Dot Product

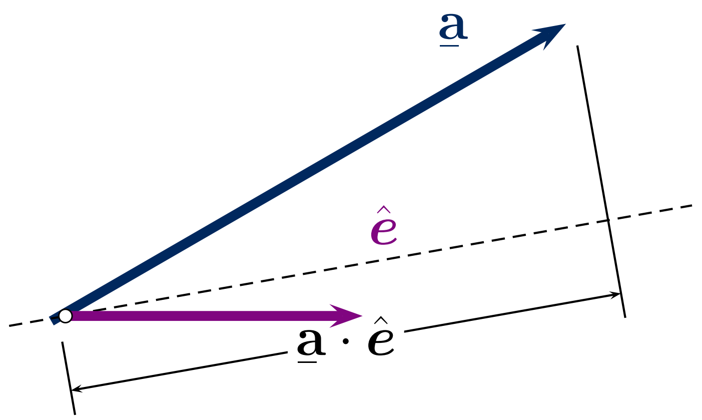
The vector dot product operates on two vectors, giving a scalar. The dot product of perpendicular vectors is zero, and defined so that

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} = 1, \qquad \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\imath}} = 0.$$

With $\underline{\mathbf{a}} = x_A\,\hat{\boldsymbol{\imath}} + y_A\,\hat{\boldsymbol{\jmath}}$ and $\underline{\mathbf{b}} = x_B\,\hat{\boldsymbol{\imath}} + y_B\,\hat{\boldsymbol{\jmath}}$, then

$$\begin{aligned}\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} &= \left(x_A\,\hat{\boldsymbol{\imath}} + y_A\,\hat{\boldsymbol{\jmath}}\right) \cdot \left(x_B\,\hat{\boldsymbol{\imath}} + y_B\,\hat{\boldsymbol{\jmath}}\right), \\ &= x_A\,x_B\,(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}) + (x_A\,y_B + y_A\,x_B)\,(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}) + y_A\,y_B\,(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}}), \\ &= x_A\,x_B + y_A\,y_B = \underline{\mathbf{b}} \cdot \underline{\mathbf{a}}.\end{aligned}$$





If the vector $\hat{e}$ has unit length, then $\underline{\mathbf{a}} \cdot \hat{e}$ is the *projection* of $\underline{\mathbf{a}}$ in the $\hat{e}$ direction.

The magnitude of a vector can be expressed as

$$\|\underline{\mathbf{a}}\| = \sqrt{(\underline{\mathbf{a}} \cdot \underline{\mathbf{a}})} = \sqrt{x_A^2 + y_A^2}.$$

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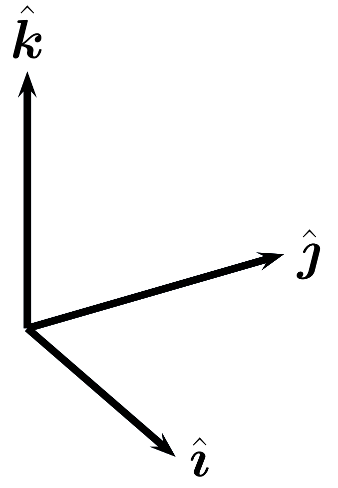
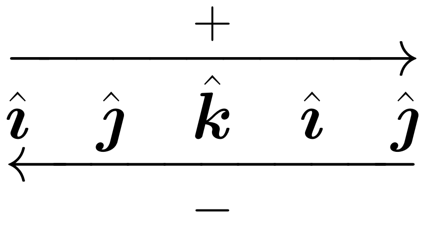
Cross Product

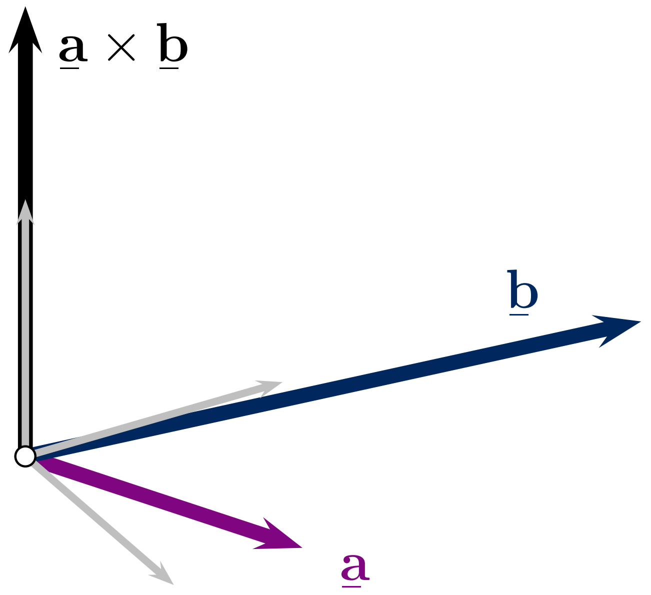
Vector cross products operate on two vectors, producing a third vector perpendicular to the two, defined as



$$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \mathbf{0}, \qquad \hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}, \qquad \hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}} = -\hat{\boldsymbol{\jmath}},$$
$$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}}, \qquad \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \mathbf{0}, \qquad \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}},$$
$$\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\imath}}, \qquad \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = \mathbf{0}.$$



The cross product of two parallel vectors is zero.



With $\underline{\mathbf{a}} = x_A\,\hat{\boldsymbol{\imath}} + y_A\,\hat{\boldsymbol{\jmath}}$ and $\underline{\mathbf{b}} = x_B\,\hat{\boldsymbol{\imath}} + y_B\,\hat{\boldsymbol{\jmath}}$

$$\begin{aligned}
\underline{\mathbf{a}} \times \underline{\mathbf{b}} &= \left(x_A\,\hat{\boldsymbol{\imath}} + y_A\,\hat{\boldsymbol{\jmath}}\right) \times \left(x_B\,\hat{\boldsymbol{\imath}} + y_B\,\hat{\boldsymbol{\jmath}}\right), \\
&= x_A\,x_B\left(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}}\right) + x_A\,y_B\left(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}\right) \\
&\quad + y_A\,x_B\left(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}}\right) + y_A\,y_B\left(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}}\right), \\
&= \left(x_A\,y_B - y_A\,x_B\right)\hat{\boldsymbol{k}} = -\underline{\mathbf{b}} \times \underline{\mathbf{a}}.
\end{aligned}$$

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