Problem 1: (30 pts.)
As illustrated in the figure, the projectile, subject to only the influence of gravity, is launched with initial speed \( v_0 \), an angle \( \theta_0 = 60^\circ \), and a distance \( \ell = 10 \) m from the edge. For what range of initial speeds will the mass will reach the lower platform?

Solution:
a) As shown in the figure, the position of the projectile \( P \) with respect to the firing point \( O \) is \( r_{PO} = x(t) \hat{i} + y(t) \hat{j} \), so that \( x(t) \) measures the horizontal distance travelled while \( y(t) \) measures the vertical displacement of \( P \). The initial position of the projectile is \( r_{PO}(0) = x(0) \hat{i} + y(0) \hat{j} = 0 \), while the initial velocity is

\[
\vec{v}_P(0) = \dot{x}(0) \hat{i} + \dot{y}(0) \hat{j} = v_0 \cos \theta \hat{i} + v_0 \sin \theta \hat{j}.
\]

The acceleration of the projectile is then determined to be

\[
\vec{a}_P = \ddot{x} \hat{i} + \ddot{y} \hat{j}.
\]

A suitable free-body diagram for the projectile is shown to the right, where only the force due to gravity acts on \( P \). Therefore the equations of motion can be written as

\[
\sum \vec{F} = m \vec{a}_P,
\]

\[
-mg \hat{j} = m (\ddot{x} \hat{i} + \ddot{y} \hat{j}).
\]

or taking components in the \( \hat{i} \) and \( \hat{j} \) directions

\[
\ddot{x} = 0, \quad \ddot{y} = -g.
\]
Using the initial conditions
\[
\begin{align*}
    x(0) &= 0, & \dot{x}(0) &= v_0 \cos \theta, \\
    y(0) &= 0, & \dot{y}(0) &= v_0 \sin \theta,
\end{align*}
\]
these equations may be solved to yield
\[
\begin{align*}
    x(t) &= (v_0 \cos \theta) t, \\
    y(t) &= -\frac{g}{2} t^2 + (v_0 \sin \theta) t.
\end{align*}
\]
Notice that we can eliminate \( t \) from these two equations to directly relate \( x \) and \( y \) as
\[
y = -\frac{g}{2(v_0 \cos \theta)^2} x^2 + \frac{\sin \theta}{\cos \theta} x = \frac{(2 v_0^2 \sin \theta \cos \theta - g x) x}{2 v_0^2 \cos^2 \theta}.
\]
At the minimum initial speed \( v_0 = v_{\text{min}} \) the projectile just clears upper edge at \((x, y) = (\ell, 0)\), so that
\[
0 = \frac{(2 v_{\text{min}}^2 \sin \theta \cos \theta - g \ell) \ell}{2 v_{\text{min}}^2 \cos^2 \theta}.
\]
Solving for \( v_{\text{min}} \)
\[
v_{\text{min}} = \sqrt{\frac{g \ell \sin \theta}{2 \cos \theta}} = \sqrt{\frac{g \ell}{\sin 2 \theta}}.
\]
Likewise, at the maximum initial speed the projectile reaches the lower edge at \((x, y) = (3\ell/2, -\ell/2)\), so that
\[
-\frac{\ell}{2} = \frac{(2 v_{\text{max}}^2 \sin \theta \cos \theta - \frac{3g\ell}{2}) \frac{3\ell}{2}}{2 v_{\text{max}}^2 \cos^2 \theta},
\]
and solving for \( v_{\text{max}} \)
\[
v_{\text{max}} = \sqrt{\frac{9 g \ell}{4(3 \cos \theta \sin \theta + \cos^2 \theta)}} = \sqrt{\frac{9 g \ell}{2(1 + \cos 2 \theta + 3 \sin 2 \theta)}}.
\]
Finally, the mass reaches the lower platform for \( v_{\text{min}} < v_0 < v_{\text{max}} \), or, with the values given above
\[
10.64 \text{ m/s} < v_0 < 11.94 \text{ m/s}.
\]
Problem 2: (20 pts.)
A disk rotates with constant angular velocity \( \dot{\theta} = \omega = 0.25 \text{ rad/s} \). The particle \( P \) moves along a radial direction relative to the disk at constant rate \( \dot{r} = v = 1 \text{ m/s} \).

a) Find the acceleration of the particle when \( r = 0.75 \text{ m} \) in terms of radial and transverse directions (\( \hat{e}_r \) and \( \hat{e}_\theta \)).

b) What force must act on the particle at this instant to achieve this motion?

Solution:

a) In terms of polar coordinates, the acceleration of the particle can be written as

\[
\hat{\mathbf{a}}_{\text{p}} = \left( \ddot{r} - r \dot{\theta}^2 \right) \hat{e}_r + \left( r \ddot{\theta} + 2 \dot{r} \dot{\theta} \right) \hat{e}_\theta.
\]

For this system, we find that

\[
\dot{r} = v, \quad \ddot{r} = 0, \quad \dot{\theta} = \omega, \quad \ddot{\theta} = 0,
\]

so that the acceleration of \( P \) becomes

\[
\hat{\mathbf{a}}_{\text{p}} = (-r \omega^2) \hat{e}_r + (2v \omega) \hat{e}_\theta = \left( -\frac{3}{64} \text{ m/s}^2 \right) \hat{e}_r + \left( \frac{1}{2} \text{ m/s}^2 \right) \hat{e}_\theta.
\]

b) Linear momentum balance implies that \( \sum \mathbf{F} = m \hat{\mathbf{a}}_{\text{p}} \). Since the acceleration is known from above, the force required to act on the particle must be

\[
\sum \mathbf{F} = m \hat{\mathbf{a}}_{\text{p}} = m \left[ \left( -\frac{3}{64} \text{ m/s}^2 \right) \hat{e}_r + \left( \frac{1}{2} \text{ m/s}^2 \right) \hat{e}_\theta \right],
\]

where \( m \) is the mass of the particle.
Problem 3: (25 pts.)
A particle of mass \( m \) moves along a frictionless circular bar of radius \( r \), and is subject to gravity and a horizontal force \( F \). i.

a) What is the angular acceleration \( \ddot{\theta} \) of the particle?

b) Find the magnitude of the force that the bar applies to the particle.

Note that your answers may (or may not) depend on \( \theta \) and \( \dot{\theta} \).

Solution:

a) With the coordinate \( \theta \) as defined above, the acceleration of the mass \( P \) can be written as

\[
\sum F = m \vec{a}_P = \left( -r \dot{\theta}^2 \right) \hat{e}_r + \left( r \ddot{\theta} \right) \hat{e}_\theta.
\]

A free-body diagram for the mass is shown to the right. In addition to the gravitational force and the applied force, a normal force in the \( \hat{e}_r \) direction represents the interaction between the frictionless hoop and the mass. Therefore the equations of motion can be written as

\[
\sum F = m \vec{a}_P, \quad F \hat{i} - mg \hat{j} + N \hat{e}_r = m \left( \left( -r \dot{\theta}^2 \right) \hat{e}_r + \left( r \ddot{\theta} \right) \hat{e}_\theta \right).
\]

Relating the (\( \hat{i}, \hat{j} \)) directions to the (\( \hat{e}_r, \hat{e}_\theta \)) directions yields

\[
\hat{i} = \cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta, \quad \hat{j} = \sin \theta \hat{e}_r + \cos \theta \hat{e}_\theta.
\]

Therefore, to determine the angular acceleration of the particle, we take the components of the this equation in the \( \hat{e}_\theta \) direction, which eliminates the unknown normal force \( N \). This becomes

\[
-F \sin \theta - mg \cos \theta = m r \ddot{\theta}.
\]

Finally, solving for \( \ddot{\theta} \)

\[
\ddot{\theta} = -\frac{F \sin \theta + mg \cos \theta}{m r}.
\]

b) The force between the bar and the particle, described as \( N \) in the \( \hat{e}_r \) direction, can be determined from the equations of motion in the \( \hat{e}_r \) direction as

\[
F \cos \theta - mg \sin \theta + N = -m r \ddot{\theta}^2.
\]
so that solving for \( N \) yields
\[
N = mg \sin \theta - F \cos \theta - m r \dot{\theta}^2.
\]

**Problem 4:** (25 pts.)
The two blocks of mass \( m \) are connected by a cable. The left surface is rough with a coefficient of friction \( \mu = 0.25 \) while the right surface is smooth \( (\mu_{\text{right}} = 0) \). The left surface is inclined at an angle \( \phi \) and the angle between the surfaces is 90°.

a) for what range of the angle \( \phi \) do the blocks slip?

b) assuming the blocks do slip, what are their accelerations?

**Solution:**
As shown in the figure above, we measure the displacement of each block by the coordinate \( x(t) \) and identify the directions \( (\hat{\mathbf{i}}_2, \hat{\mathbf{j}}_2) \) aligned with the surface. Therefore the acceleration of the block resting on the rough surface is \( \hat{\mathbf{x}} a_{G\ell} = \ddot{x} \hat{i}_2 \), while the acceleration of the block on the smooth surface is \( \hat{\mathbf{x}} a_G = -\ddot{x} \hat{j}_2 \).

Free-body diagrams for each block are shown to the right. Notice that a force due to friction acts only the block resting on the rough surface, although both blocks are subject to forces normal to their respective surfaces.

On the left block the equations of motion can therefore be written as

\[
\sum F_{\ell} = m \hat{\mathbf{x}} a_{G\ell},
\]
\[
T \hat{i}_2 + f_{\ell} \hat{i}_2 + N_{\ell} \hat{j}_2 - mg \hat{j} = m (\ddot{x} \hat{i}_2),
\]

while on the right block

\[
\sum F_r = m \hat{\mathbf{x}} a_G,
\]
\[
T \hat{j}_2 + N_r \hat{i}_2 - mg \hat{j} = m (-\ddot{x} \hat{j}_2).
\]

The direction of the gravitational force \( \hat{j} \) can be written in terms of \( (\hat{i}_2, \hat{j}_2) \) as
\[
\hat{j} = \sin \phi \hat{i}_2 + \cos \phi \hat{j}_2.
\]

Finally, taking components of these equations in terms of \( (\hat{i}_2, \hat{j}_2) \), we obtain the four scalar equations
\[
T + f_{\ell} - mg \sin \phi = m \ddot{x}, \quad N_{\ell} - mg \cos \phi = 0,
\]
\[
N_r - mg \sin \phi = 0, \quad T - mg \cos \phi = -m \ddot{x}.
\]
We can immediately solve for the unknown normal forces acting on the two blocks as

\[ N_L = mg \cos \theta, \quad N_r = mg \sin \theta. \]

Eliminating the unknown tension yields

\[ 2m \ddot{x} = mg (\cos \phi - \sin \phi) + f_L, \]

where \( f_L \) is the (as yet) unknown friction force acting on the right block. If the blocks slip \( \dot{x} \neq 0 \) and

\[ f_{L,\text{slip}} = -\mu N_L \text{sgn}(\dot{x}) = -\mu mg \cos \phi \text{sgn}(\dot{x}), \]

while if the blocks are stationary then \( \dot{x} = 0 \) and

\[ |f_L| \leq \mu N_L = \mu mg \cos \phi. \]

(a) The blocks slip provided the friction force \( f_L \) is unable to maintain static equilibrium, which implies that for equilibrium

\[ |f_{L,\text{eq}}| = |mg (\cos \phi - \sin \phi)| \leq \mu mg \cos \phi, \]

or

\[ -\mu mg \cos \phi \leq mg (\cos \phi - \sin \phi) \leq \mu mg \cos \phi. \]

This can be solved for \( \phi \) so that static equilibrium requires

\[ 1 - \mu \leq \tan \phi \leq 1 + \mu. \]

Therefore, the system slides for \( \phi \) outside this interval, which for these parameter values yields

\[ |\phi| < 36.9^\circ, \quad \text{and} \quad 51.3^\circ < |\phi| < 90^\circ. \]

(b) If the blocks slip then the friction force is \( f_L = -\mu mg \cos \phi \text{sgn}(\dot{x}) \), so that the \( \ddot{x} \) may be determined to be

\[ \ddot{x} = \frac{g}{2} ((1 - \mu \text{sgn}(\dot{x})) \cos \phi - \sin \phi). \]