Problem 1: (30 pts.)
The point mass \( m \) shown to the right is supported on the rotating bar by a spring with unstretched length \( \ell \) and stiffness \( k \). If the bar is rotating at constant angular speed \( \dot{\theta} = \omega \):

a) find the acceleration of the mass relative to the bar (in the radial direction) as a function of \( r(t) \), the radial displacement;

b) when the mass is in equilibrium relative to the bar (in the radial direction), what is the steady-state compression of the spring?

Solution:

a) In addition to the coordinates \( r(t) \) and \( \theta(t) \) as shown in the figure, we also identify the stretch in the spring as \( \delta(t) \), related to \( z(t) \) as

\[
\delta(t) = d - \ell - r(t).
\]

The directions \( \hat{e}_r \) and \( \hat{e}_\theta \) are radial and tangent to the rotating bar.

Neglecting gravity, a free-body diagram for this system is shown to the right. The force \( N \hat{e}_\theta \) represents the force from the bar applied to the point mass while the force from the spring is \( k \delta(t) \hat{e}_r \). Finally, in terms of the coordinates \( r(t) \) and \( \theta(t) \), the acceleration of the mass can be written as

\[
\vec{a}_G = (\ddot{r}(t) - r(t) \omega^2) \hat{e}_r + (2 \dot{r}(t) \omega) \hat{e}_\theta,
\]

where we have made use of the constant angular velocity of the bar, so that \( \ddot{\theta}(t) = 0 \).

Applying momentum balance to this system

\[
\sum \vec{F} = (k \delta(t)) \hat{e}_r + (N(t)) \hat{e}_\theta = m \left( (\ddot{r}(t) - r(t) \omega^2) \hat{e}_r + (2 \dot{r}(t) \omega) \hat{e}_\theta \right) = m \vec{a}_G.
\]
Thus, taking the component in the $\hat{e}_r$ direction
\[ k \delta(t) = m \left( \ddot{r}(t) - r(t) \omega^2 \right). \]

Finally, solving for the acceleration along the bar, we obtain
\[ \ddot{r}(t) = \frac{k}{m} \left( d - \ell - r(t) \right) + r(t) \omega^2, \]
\[ = \frac{k(d - \ell)}{m} - \frac{k - m \omega^2}{m} r(t). \]

b) The equilibrium position of the mass relative to the bar implies that $\ddot{r}(t) \equiv 0$, so that the above equation can be solved to yield
\[ r_{eq} = \frac{k(d - \ell)}{k - m \omega^2}. \]

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**Problem 2:** (20 pts.)

The two blocks shown to the right, each of mass $m = 2 \text{ kg}$, rest on a rough surface with coefficient of friction $\mu = 0.25$. The initial velocity of the left block is $v \hat{i}$, with $v = 1.50 \text{ m/s}$ while the second block, located a distance $d = 0.25 \text{ m}$ away is initially at rest. If the coefficient of restitution between the blocks is $e = 0.80$, find the distance the right block slides after the impact before coming to rest.

**Solution:**

The motion of this system can be divided into three phases. In the first, the block $A$ slides to the right. The second phase describes the impact between blocks $A$ and $B$, while the final phase describes the motion of block $B$ as it comes to rest. The first and third phase can be described using the work-energy formulation while the collision is best described using an impulse-momentum formulation.

Because the blocks are identical, a free-body diagram of either block is shown to the right. Since the motion is only in the $\hat{i}$ direction, only the force in the $\hat{i}$ direction contributes to the work.

The work-energy formulation for either sliding block yields
\[ \int_0^{x(t)} (-\mu m g \hat{i}) \cdot (dx \hat{i}) = \frac{m}{2} (\dot{x}^2(t) - \dot{x}^2(0)), \]
so that the velocity of block $A$ at the impact, $\dot{x}_{A,1}$ is determined from
\[ -\mu m g d = \frac{m}{2} (\dot{x}^2_{A,1} - v^2). \]

Solving for the velocity of block $A$ just before the impact, $\dot{x}_{A,1}$
\[ \dot{x}_{A,1} = \sqrt{v^2 - 2 \mu g d}. \]
Across the impact conservation of momentum holds

\[ m_A \ddot{x}_{A,1} + m_B \ddot{x}_{B,1} = m_A \ddot{x}_{A,2} + m_B \ddot{x}_{B,2}, \quad \rightarrow \ddot{x}_{A,1} = \ddot{x}_{A,2} + \ddot{x}_{B,2}. \]

Together with the definition of the coefficient of restitution

\[ e = \frac{\dot{x}_{B,2} - \dot{x}_{A,2}}{\dot{x}_{A,1} - \dot{x}_{B,1}}, \quad \rightarrow e \dot{x}_{A,1} = \dot{x}_{B,2} - \dot{x}_{A,2}. \]

Therefore, solving for the velocity of block \( B \) after the collision, \( \ddot{x}_{B,2} \)

\[ \ddot{x}_{B,2} = \frac{1 + e}{2} \dot{x}_{A,1}. \]

Over the final interval of the motion we again use a work-energy formulation for block \( B \). If the distance traveled by block \( B \) before coming to rest is \( \delta \), then \( \dot{x}_{B,3} = 0 \) and using the relation obtained above

\[ -\mu m g \delta = \frac{m}{2} \ddot{x}_{B,2}^2, \]

and solving for \( \delta \)

\[ \delta = \frac{1}{2 \mu g} \ddot{x}_{B,2}^2, \]

\[ = \frac{(1 + e)^2}{8 \mu g} \dot{x}_{A,1}^2, \]

\[ = \frac{(1 + e)^2}{8 \mu g} \left( v^2 - 2 \mu g d \right) = \frac{(1 + e)^2}{4} \left( \frac{v^2}{2 \mu g} - d \right). \]

Finally, with the supplied values, we find that block \( B \) slides a distance \( \delta = 0.17 \) m before coming to rest.

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**Problem 3:** (30 pts.)

In the system shown to the right, the two masses \((m = 1 \text{ kg})\) are connected by a cable wrapped around a massless pulley. If the system is released from rest, with the spring \((k = 250 \text{ N/m})\) compressed by a distance \( d = 0.15 \text{ m} \), from its unstretched position:

- a) find the velocity of the mass on the right when the spring is unstretched;

- b) what is the maximum velocity of the mass on the left?
Solution:

We choose a work-energy formulation to describe the motion of this system. Specifically, all forces are conservative so that the total energy of the system is conserved

\[ T_i + V_i = T_f + V_f, \]

where \( T \) and \( V \) are the kinetic and potential energies of the system.

The displacement of each mass relative to its position when the spring is unstretched is described by \( x \), so that the kinetic and potential energies can be written as

\[ T = \frac{1}{2} (m) \dot{x}^2 + \frac{1}{2} (4m) \dot{x}^2 = \frac{1}{2} (5m) \dot{x}^2, \]
\[ V = \frac{1}{2} k x^2 + m g x - (4m) g x = \frac{1}{2} k x^2 - (3m) g x. \]

Notice that the potential energy has been taken relative to the configuration of the system when the spring is unstretched.

For an initial compression in the spring of \( d \), so that \( x_i = -d \), conservation of energy yields

\[ \frac{k}{2} d^2 + 3 m g d = \frac{5m}{2} \dot{x}^2 + \frac{k}{2} x^2 - 3 m g x. \]

a) When the spring is unstretched, \( x_f = 0 \) and solving the above for \( \dot{x}_f \) yields

\[ \dot{x}_f = \sqrt{\frac{k d^2 + 6 m g d}{5 m}}. \]

Using the provided parameter values, \( \dot{x}_f = 1.70 \) m/s.

b) The maximum kinetic energy of the system occurs when the potential energy is minimized. Thus finding the location \( x = x_m \) at which this occurs, \( V_{\text{min}} \)

\[ \frac{dV}{dx} = k x_m - 3 m g = 0, \quad \rightarrow x_m = \frac{3 m g}{k}. \]

Therefore, at this position the velocity of the masses can be determined to be

\[ \dot{x}_m = \sqrt{\frac{(k d)^2 + 6 (m g) (k d) + 9 (m g)^2}{5 k m}}, \]
\[ = \sqrt{\frac{k d + 3 m g}{5 k m}}. \]

Using the numerical values for this problem, \( \dot{x}_m = 1.89 \) m/s.