Problem 1: (20 pts.)
The two blocks shown to the right are constrained to move in orthogonal slots and are connected by a rod of length $\ell = 10\, \text{cm}$. If the velocity of $A$ is constant, so that $\dot{\mathbf{v}}_A = v \mathbf{j}$, find the velocity and acceleration of block $B$, as a function of $v$, when $x = 6\, \text{cm}$.

Solution:

a) We define the angle $\theta$ of the link and the displacement of block as $z$, as shown in the figure. Thus

$$
\dot{\mathbf{e}}_1 = \cos \theta \mathbf{i} - \sin \theta \mathbf{j}, \quad \dot{\mathbf{e}}_2 = \sin \theta \mathbf{j} + \sin \theta \mathbf{j}.
$$

In addition, at this instant $\theta = 36.87^\circ$ and $z = 8\, \text{cm}$. Given the velocity of $A$, the velocity of $B$ can be written as

$$
\dot{\mathbf{x}}_{B} = \dot{\mathbf{x}}_{A} + \mathbf{\omega}_{BA} \times \mathbf{r}_{BA}.
$$

which reduces to

$$
\dot{\mathbf{x}}_{B} = (v \mathbf{j}) + \left(-\dot{\theta} \mathbf{k}\right) \times (\ell \mathbf{e}_1),
$$

$$
= \left(-\ell \dot{\theta} \sin \theta \right) \mathbf{i} + \left(v - \ell \dot{\theta} \cos \theta \right) \mathbf{j}.
$$

The velocity of block $B$ only lies in the $\mathbf{i}$ direction so that the $\mathbf{j}$ component must vanish, which implies that

$$
\dot{\theta} = \frac{v}{\ell \cos \theta}.
$$
Returning this to the velocity
\[ \dot{\mathbf{v}}_B = -\ell \sin \theta \left( \frac{v}{\ell \cos \theta} \right) \hat{i} = -v \tan \theta \hat{i}. \]

Finally, at this instant \( \tan \theta = 3/4 \), so that \( \dot{\mathbf{v}}_B = \frac{3v}{4} \hat{i} \).

b) The acceleration of the lower block can be written as
\[ \dot{\mathbf{a}}_B = \dot{\mathbf{a}}_A + \mathbf{a}_{h/x} \times \mathbf{r}_{BA} + \omega_{h/x} \times (\omega_{h/x} \times \mathbf{r}_{BA}). \]

The acceleration of \( A \) vanishes, so that \( \dot{\mathbf{a}}_A = 0 \), and with the angular acceleration of the bar identified as \( \omega_{h/x} = -\theta \dot{k} \), this reduces to
\[ \dot{\mathbf{a}}_B = \left( -\ell \ddot{\theta} \sin \theta - \ell \dot{\theta}^2 \cos \theta \right) \hat{i} + \left( -\ell \ddot{\theta} \cos \theta + \ell \dot{\theta}^2 \sin \theta \right) \hat{j}. \]

As for the velocity, the component of the acceleration in the \( \hat{j} \) direction must vanish, so that
\[ \ddot{\theta} = \dot{\theta}^2 \tan \theta. \]

Finally, returning to the acceleration of \( B \)
\[ \dot{\mathbf{a}}_B = -\ell \dot{\theta}^2 \cos \theta \hat{i} = -v^2 \frac{\ell}{\cos^3 \theta} \hat{i}. \]

Finally, at this instant
\[ \dot{\mathbf{a}}_B = -\frac{25v^2}{64} \hat{i}. \]

**Problem 2: (20 pts.)**

A sled of mass \( m_1 \) is pulled to the right by a vehicle of mass \( m_2 = 1000 \text{ kg} \) along a rough surface with coefficient of friction \( \mu = 0.25 \). In terms of the mass of the sled \( m_1 \), find the maximum driving force that can be applied from the rear wheels so that contact remains between the front wheels of the vehicle and the surface, i.e., the vehicle does not tip over. Assume that the width of the sled is such that it never tips. Note that all distances are given in terms of \( h = 1.50 \text{ m} \).

**Solution:**

We define the displacement of the system as \( x \) as described in the figure. Therefore the acceleration of both the vehicle and the sled can be written as \( \dot{\mathbf{a}}_G = \dot{\mathbf{a}}_P = \dot{x} \hat{i} \).
Then, free body diagrams for both the sled and the vehicle can be constructed as shown to the right. The tension in the cable connecting the sled and the vehicle has (unknown) magnitude $T$ while the driving force applied at the rear wheels is $f_d \hat{i}$. Notice that the normal force acting on both wheels has been included. However, when the vehicle is just about to tip the normal force at the front wheels vanishes, so that $N_3 = 0$. This is the condition that determines the maximum driving force $f_{d,\text{max}}$.

The equations of motion will be developed by applying linear momentum balance separately on both the sled and the vehicle, and angular momentum balance about $G$ on the vehicle. Because the sled is assumed to never tip, we can safely neglect angular momentum balance on this component.

Linear momentum balance on the sled provides

$$\sum \mathbf{F} = (T - \mu N_1) \hat{i} + (N_1 - m_1 g) \hat{j} = m_1 \ddot{x} \hat{i} = m_1 \mathbf{a}_x,$$

while linear and angular momentum balance on the vehicle yield

$$\sum \mathbf{F} = (-T + f_d) \hat{i} + (N_2 + N_3 - m_2 g) \hat{j} = m_2 \ddot{x} \hat{i} = m_2 \mathbf{a}_x,$$

$$\sum \mathbf{M}_G = \left( T h + f_d \frac{h}{2} + N_3 (2 h) - N_2 h \right) \hat{k} = 0.$$

From the linear momentum balance equations in the $\hat{j}$ direction we find that $N_1 = m_1 g$ and $N_2 = m_2 g - N_3$, so that when the vehicle is about to tip $N_3 = 0$, and $N_2 = m_2 g$. Then, the remaining scalar equations are

$$T - \mu m_1 g = m_1 \ddot{x},$$

$$-T + f_{d,\text{max}} = m_2 \ddot{x},$$

$$\frac{h}{2} (2 T + f_{d,\text{max}} - 2 m_2 g) = 0.$$

Finally, solving these for $f_{d,\text{max}}$ we find that

$$f_{d,\text{max}} = \frac{2 m_2 g \left( (1 - \mu) m_1 + m_2 \right)}{3 m_1 + m_2} = \left[ \frac{0.75 m_1 + 1000 \text{ kg}}{3 m_1 + 1000 \text{ kg}} \right] 19620 \text{ N}.$$

Although this was not part of the problem statement, from the above equations of motion the tension in the cable and the acceleration of the system when the vehicle is on the verge of tipping can be solved as

$$T = \frac{(2 + \mu) m_1 m_2 g}{3 m_1 + m_2}, \quad \ddot{x} = \frac{(2 m_2 - 3 \mu m_1) g}{3 m_1 + m_2}.$$
Note that the acceleration of the block is positive and the block continues to slide to the right provided \( m_1 < (2 m_2)/(3 \mu) \).

**Problem 3:** (20 pts.)
The two masses are supported by cables of equal length \( \ell = 40 \text{ cm} \). The left mass is released from rest in the horizontal position, falls down and impacts the second mass, with \( m_2 = 2 \text{ kg} \), which is initially stationary in its vertical equilibrium position. Find the coefficient of restitution \( e \) between the two masses and the mass of the left mass \( m_1 \) so that after the impact the left mass remains stationary and the right mass swings through a maximum angle of \( \phi = 60^\circ \).

Note: at \( t = 0 \)
\[
\phi_1(0) = -\pi/2 \\
\phi_2(0) = 0 
\]
and at \( t = t_3 \)
\[
\phi_1(t_3) = 0 
\]

**Solution:**
The response of this system can be described in three phases, in which energy is conserved as the left pendulum falls \((t_0 < t < t_1)\), conservation of momentum during the impact \((t_1 = t^-_{\text{impact}} < t < t^+_{\text{impact}} = t_2)\), and conservation of energy again as the right pendulum swings up \((t_2 < t < t_3)\).

With the coordinates \( \phi_1 \) and \( \phi_2 \) defined as shown in the figure, the speed of each mass can be expressed as
\[
v_1 = \ell \dot{\phi}_1, \quad v_2 = \ell \dot{\phi}_2. 
\]

Finally, the speed of mass \( i \) at time \( t_j \) is denoted as \( v_{i,j} = v_i(t_j) \), so that
\[
v_{1,0} = v_{2,0} = 0, \quad v_{2,1} = 0, \quad v_{1,2} = 0, \quad v_{1,3} = v_{2,3} = 0. 
\]

In general, the kinetic and potential energy of this systems can be defined as
\[
\mathcal{T} = \frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2, \quad \mathcal{V} = m_1 g \ell \ (1 - \cos \phi_1) + m_2 g \ell \ (1 - \cos \phi_2), 
\]
where the potential energy is assumed to vanish at the lower position of the pendula. Therefore the energy \( \mathcal{E} = \mathcal{T} + \mathcal{V} \) at each time can be expressed as
\[
\mathcal{E}_0 = \mathcal{T}_0 + \mathcal{V}_0 = m_1 g \ell, \quad \mathcal{E}_2 = \mathcal{T}_2 + \mathcal{V}_2 = \frac{m_2}{2} v_{2,2}^2, \\
\mathcal{E}_1 = \mathcal{T}_1 + \mathcal{V}_1 = \frac{m_1}{2} v_{1,1}^2, \quad \mathcal{E}_3 = \mathcal{T}_3 + \mathcal{V}_3 = m_2 g \ell \ (1 - \cos 60^\circ) = m_2 g \frac{\ell}{2}. 
\]

Therefore across the initial and final stages of the motion, during which energy is conserved
\[
v_{1,2} = \sqrt{2 g \ell}, \quad v_{2,2} = \sqrt{g \ell}. 
\]

During the impact the linear momentum is conserved, so that
\[
m_1 v_{1,1} + m_2 v_{2,1} = m_1 v_{1,2} + m_2 v_{2,2} \quad \rightarrow \quad m_1 v_{1,1} = m_2 v_{2,2}. 
\]
In addition, the coefficient of restitution is defined as

\[ e = \frac{v_{2,2} - v_{1,2}}{v_{1,1} - v_{2,1}} \quad \rightarrow \quad e = \frac{v_{2,2}}{v_{1,1}} \]

Substituting in for the values of \( v_{2,2} \) and \( v_{1,1} \) found earlier

\[ e = \frac{\sqrt{g \ell}}{\sqrt{2 g \ell}} = \frac{1}{\sqrt{2}} \quad m_1 = m_2 = \frac{m_2 \sqrt{g \ell}}{\sqrt{2}} = \sqrt{2} \text{ kg} \]

**Problem 4:** (20 pts.)

In the mechanism shown, the inner arm (length \( \ell = 25 \text{ cm} \)) spins at a constant angular speed \( \omega_2 = \theta_2 = 10 \text{ rad/s} \), while the disk, attached to the arm at \( A \), spins at constant rate \( \omega_3 = \theta_3 = 5 \text{ rad/s} \) with respect to the ground. Finally, the particle \( P \) with mass \( m = 1.25 \text{ kg} \) slides in a smooth slot on the disk and is attached to a spring (of stiffness \( k = 50 \text{ N/m} \)). Assume that the spring is unstretched when the particle is at the center of the disk. At this instant, \( \theta_2 = 60^\circ \), \( \theta_3 = 45^\circ \), and \( x = 8 \text{ cm} \):

a) Determine the acceleration of \( A \), the center of the disk, with respect to the ground;

b) Determine the acceleration of \( P \) with respect to the disk at this instant, that is, relative to a frame of reference fixed in the disk in the direction of the slot.

*Hint: you must use linear momentum balance on the particle.*

**Solution:**

The above directions can be related as

\[ \hat{i}_2 = \cos \theta_2 \hat{i} + \sin \theta_2 \hat{j}, \quad \hat{i}_3 = \cos \theta_3 \hat{i} + \sin \theta_3 \hat{j}, \quad \hat{j}_2 = -\sin \theta_2 \hat{i} + \cos \theta_2 \hat{j}, \quad \hat{j}_3 = -\sin \theta_3 \hat{i} + \cos \theta_3 \hat{j}. \]

a) In terms of the coordinates provided and the directions defined above, the acceleration of \( A \) can be expressed as

\[ \vec{x} a_A = \vec{\omega}_{\beta/\xi} \times (\vec{\omega}_{\beta/\xi} \times \vec{x}_{AO}) = \left( \dot{\theta}_2 \hat{k} \right) \times \left( \left( \dot{\theta}_2 \hat{k} \right) \times (\ell \hat{i}_2) \right) = -\ell \dot{\theta}_2^2 \hat{i}_2. \]

Therefore, at this instant

\[ \vec{x} a_A = -(25 \text{ m/s}) \hat{i}_2 = -(12.5 \text{ m/s}) \hat{i} - (21.6 \text{ m/s}) \hat{j}. \]

b) The acceleration of \( P \) with respect to the disk is

\[ \vec{x} a_P = \vec{x} \hat{i}_3, \]
which can be related to the response of the system through linear momentum balance, 
\[ \sum \mathbf{F} = m \overset{\cdot}{\mathbf{a}}_P, \]
with \[ \overset{\cdot}{\mathbf{a}}_P = \overset{\cdot}{\mathbf{a}}_A + \overset{\cdot}{\mathbf{a}}_P + \overset{\cdot}{\mathbf{a}}_D/F \times \mathbf{r}_{PA} + (\overset{\cdot}{\mathbf{a}}_D/F \times \mathbf{r}_{PA}) + 2 \overset{\cdot}{\mathbf{a}}_D/F \times \overset{\cdot}{\mathbf{v}}_P. \]

In this the angular acceleration of the disk vanishes and the velocity of \( P \) with respect to the disk is \( \overset{\cdot}{\mathbf{v}}_P = \dot{x} \hat{i}_3 \), so that the acceleration of \( P \) becomes

\[ \overset{\cdot}{\mathbf{a}}_P = -\ell \dot{\theta}_2^2 \hat{i}_2 + \ddot{x} \hat{i}_3 + \left( \left( \dot{\theta}_3 \hat{k} \right) \times (x \hat{i}_3) \right) + 2 \dot{\theta}_3 \hat{k} \times (\dot{x} \hat{i}_3), \]

\[ = -\ell \dot{\theta}_2^2 \hat{i}_2 + \ddot{x} \hat{i}_3 - x \dot{\theta}_3 \hat{i}_3 - 2 \dot{x} \dot{\theta}_3 \hat{j}_3. \]

An appropriate free body diagram is shown to the right, where the only forces that act on the particle are due to the spring and the normal force arising from the constrained motion within the slot.

Finally, noting that \[ \hat{i}_2 = \cos(\theta_2 - \theta_3) \hat{i}_3 + \sin(\theta_2 - \theta_3) \hat{j}_3, \]
linear momentum balance can be written in the \((\hat{i}_3, \hat{j}_3)\) directions as

\[ \sum \mathbf{F} = m \overset{\cdot}{\mathbf{a}}_P, \]

\[ -k x \hat{i}_3 + N \hat{j}_3 = m \left( \ddot{x} - x \ddot{\theta}_2^2 - \ell \dot{\theta}_2^2 \cos(\theta_2 - \theta_3) \right) \hat{i}_3 + m \left( -2 \dot{x} \dot{\theta}_3^2 - \ell \dot{\theta}_2^2 \sin(\theta_2 - \theta_3) \right) \hat{j}_3. \]

The component in the \( \hat{i}_3 \) direction may be solved for \( \ddot{x} \) as

\[ \ddot{x} = \left( \dot{\theta}_3^2 - \frac{k}{m} \right) x + \ell \dot{\theta}_2^2 \cos(\theta_2 - \theta_3). \]

For the given values we find that \( \ddot{x} = 22.95 \text{ m/s}^2 \).