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1 Kinematics

Problem 1: (20 pts.)
The two blocks shown to the right are constrained to move in orthogonal slots and are connected by a rod of length $\ell = 10$ cm. If the velocity of $A$ is constant, so that $\dot{\mathbf{r}}_A = v \mathbf{j}$, find the velocity and acceleration of block $B$, as a function of $v$, when $x = 6$ cm.

Solution:

a) We define the angle $\theta$ of the link and the displacement of block as $z$, as shown in the figure. Thus

$$\dot{\mathbf{e}}_1 = \cos \theta \mathbf{i} - \sin \theta \mathbf{j}, \quad \dot{\mathbf{e}}_2 = \sin \theta \mathbf{j} + \sin \theta \mathbf{j}.$$  

In addition, at this instant $\theta = 36.87^\circ$ and $z = 8$ cm. Given the velocity of $A$, the velocity of $B$ can be written as

$$\dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A + \omega_{\theta z} \times \mathbf{r}_{BA},$$

which reduces to

$$\dot{\mathbf{r}}_B = (v \mathbf{j}) + \left(-\hat{\theta} \mathbf{k}\right) \times (\ell \dot{\mathbf{e}}_1),$$

$$= \left(-\ell \hat{\theta} \sin \theta\right) \mathbf{i} + \left(v - \ell \hat{\theta} \cos \theta\right) \mathbf{j}.$$  

The velocity of block $B$ only lies in the $\mathbf{i}$ direction so that the $\mathbf{j}$ component must vanish, which implies that

$$\hat{\theta} = \frac{v}{\ell \cos \theta}.$$
Returning this to the velocity

\[ \mathbf{v}_B = -\ell \sin \theta \left( \frac{v}{\ell \cos \theta} \right) \mathbf{i} = -v \tan \theta \mathbf{i}. \]

Finally, at this instant \( \tan \theta = 3/4 \), so that \( \mathbf{v}_B = (3v)/4 \mathbf{i} \).

b) The acceleration of the lower block can be written as

\[ \mathbf{a}_B = \mathbf{a}_A + \alpha_{\beta/F} \times \mathbf{r}_{BA} + (\omega_{\beta/F} \times \mathbf{r}_{BA}). \]

The acceleration of \( A \) vanishes, so that \( \mathbf{a}_A = 0 \), and with the angular acceleration of the bar identified as \( \alpha_{\beta/F} = -\dot{\theta} \mathbf{k} \), this reduces to

\[ \mathbf{a}_B = \left( -\ell \dot{\theta} \sin \theta - \ell \dot{\theta}^2 \cos \theta \right) \mathbf{i} + \left( -\ell \dot{\theta} \cos \theta + \ell \dot{\theta}^2 \sin \theta \right) \mathbf{j}. \]

As for the velocity, the component of the acceleration in the \( \mathbf{j} \) direction must vanish, so that

\[ \dot{\theta} = \dot{\theta}^2 \tan \theta. \]

Finally, returning to the acceleration of \( B \)

\[ \mathbf{a}_B = -\ell \dot{\theta}^2 \cos \theta \mathbf{i} = -\frac{v^2}{\ell \cos^2 \theta} \mathbf{i}. \]

Finally, at this instant

\[ \mathbf{a}_B = -\frac{25v^2}{64} \mathbf{i}. \]

**Problem 2:** (20 pts.)

In the mechanism shown, the inner arm (length \( \ell = 25 \text{ cm} \)) spins at a constant angular speed \( \omega_2 = \dot{\theta}_2 = 10 \text{ rad/s} \), while the disk, attached to the arm at \( A \), spins at constant rate \( \omega_3 = \dot{\theta}_3 = 5 \text{ rad/s} \) with respect to the ground. Finally, the particle \( P \) with mass \( m = 1.25 \text{ kg} \) slides in a smooth slot on the disk and is attached to a spring (of stiffness \( k = 50 \text{ N/m} \)). Assume that the spring is unstretched when the particle is at the center of the disk. At this instant, \( \theta_2 = 60^\circ \), \( \theta_3 = 45^\circ \), and \( x = 8 \text{ cm} \):

a) Determine the acceleration of \( A \), the center of the disk, with respect to the ground;

b) Determine the acceleration of \( P \) with respect to the disk at this instant, that is, relative to a frame of reference fixed in the disk in the direction of the slot. *Hint: you must use linear momentum balance on the particle.*

**Solution:**
The above directions can be related as
\[ \hat{i}_2 = \cos \theta_2 \hat{i} + \sin \theta_2 \hat{j}, \quad \hat{i}_3 = \cos \theta_3 \hat{i} + \sin \theta_3 \hat{j}, \quad \hat{j}_2 = -\sin \theta_2 \hat{i} + \cos \theta_2 \hat{j}, \quad \hat{j}_3 = -\sin \theta_3 \hat{i} + \cos \theta_3 \hat{j}. \]

a) In terms of the coordinates provided and the directions defined above, the acceleration of \( A \) can be expressed as
\[ \v^{x}_A = \omega_{\beta/F} \times (\omega_{\beta/F} \times r_{A\beta}) = \left( \hat{\theta}_2 \hat{k} \right) \times \left( \left( \hat{\theta}_2 \hat{k} \right) \times (\ell \hat{i}_2) \right) = -\ell \hat{\theta}_2^2 \hat{i}_2. \]

Therefore, at this instant
\[ \v^{x}_A = -(25 \text{ m/s}) \hat{i}_2 = -(12.5 \text{ m/s}) \hat{i} - (21.6 \text{ m/s}) \hat{j}. \]

b) The acceleration of \( P \) with respect to the disk is
\[ \v^{x}_A = \ddot{x} \hat{i}_3, \]
which can be related to the response of the system through linear momentum balance, \( \sum F = m \v^{x}_a, \) with
\[ \v^{x}_a = \v^{x}_A + \dot{x} \hat{i}_3 + \left( \hat{\theta}_3 \hat{k} \right) \times \left( \left( \hat{\theta}_3 \hat{k} \right) \times (x \hat{i}_3) \right) + 2 \omega_{\beta/F} \times \v^{x}_v P. \]

In this the angular acceleration of the disk vanishes and the velocity of \( P \) with respect to the disk is \( \v^{x}_v P = \dot{x} \hat{i}_3, \) so that the acceleration of \( P \) becomes
\[ \v^{x}_a = -\ell \hat{\theta}_2^2 \hat{i}_2 + \ddot{x} \hat{i}_3 + \left( \hat{\theta}_3 \hat{k} \right) \times \left( \left( \hat{\theta}_3 \hat{k} \right) \times (x \hat{i}_3) \right) + 2 \left( \hat{\theta}_3 \hat{k} \right) \times \left( \dot{x} \hat{i}_3 \right), \]
\[ = -\ell \hat{\theta}_2^2 \hat{i}_2 + \ddot{x} \hat{i}_3 - x \hat{\theta}_3 \hat{i}_3 - 2 \dot{x} \hat{\theta}_2^2 \hat{j}_3. \]

An appropriate free body diagram is shown to the right, where the only forces that act on the particle are due to the spring and the normal force arising from the constrained motion within the slot.

Finally, noting that
\[ \dot{i}_2 = \cos(\theta_2 - \theta_3) \hat{i}_3 + \sin(\theta_2 - \theta_3) \hat{j}_3, \]
linear momentum balance can be written in the \((\hat{i}_3, \hat{j}_3)\) directions as
\[ \sum F = m \v^{x}_a, \quad -k x \hat{i}_3 + N \hat{j}_3 = m \left( \ddot{x} - x \hat{\theta}_3^2 - \ell \hat{\theta}_2^2 \cos(\theta_2 - \theta_3) \right) \hat{i}_3 + m \left( -2 \dot{x} \hat{\theta}_3^2 - \ell \hat{\theta}_2^2 \sin(\theta_2 - \theta_3) \right) \hat{j}_3. \]

The component in the \( \hat{i}_3 \) direction may be solved for \( \ddot{x} \) as
\[ \ddot{x} = \left( \hat{\theta}_3^2 - \frac{k}{m} \right) x + \ell \hat{\theta}_2^2 \cos(\theta_2 - \theta_3). \]

For the given values we find that \( \ddot{x} = 22.95 \text{ m/s}^2 \).
Problem 3:
A particle moves along a circular path of radius \( r = 10 \text{ m} \). If the speed \( v \) increases uniformly from rest with \( \dot{v} = 0.50 \text{ m/s}^2 \)

a) Find the speed and the acceleration of the particle after \( t = 4 \text{ s} \);

b) How far does the particle travel during this time?

c) If the mass of the particle is \( m = 2 \text{ kg} \), how long (in time) does the particle travel before the magnitude of the force acting on the particle is \( \| F \| = 16 \text{ N} \)?

Solution:

a) With \( \dot{v} = 0.50 \text{ m/s}^2 \) the speed of the particle when released from rest can be expressed as

\[
v(t) = (0.50 \text{ m/s}^2) t.
\]

In addition, using normal and tangential coordinates, the acceleration of the particle may be written as

\[
\mathbf{a}_P = \dot{v} \hat{e}_t + \frac{v^2}{r} \hat{e}_n = \left(0.50 \text{ m/s}^2\right) \hat{e}_t + \left((0.025 \text{ m/s}^4) t^2\right) \hat{e}_n.
\]

Therefore, at \( t = 4 \text{ s} \), the speed and acceleration of the particle are

\[
v(4 \text{ s}) = 2.00 \text{ m/s}, \quad \mathbf{a}_P(4 \text{ s}) = \left(0.50 \text{ m/s}^2\right) \hat{e}_t + \left(0.40 \text{ m/s}^2\right) \hat{e}_n.
\]

Note that the magnitude of the acceleration at \( t = 4 \text{ s} \) is \( \| \mathbf{a}_P(4 \text{ s}) \| = 0.64 \text{ m/s}^2 \).

b) The displacement \( s(t) \) of the particle can be found by integrating the speed again, so that

\[
s(t) = \left(0.25 \text{ m/s}^2\right) t^2,
\]

and at \( t = 4 \text{ s} \), the distance travelled by the particle is

\[
s(4 \text{ s}) = 4 \text{ m}.
\]

c) From momentum balance, the magnitude of the external force equals the mass times the magnitude of the acceleration, that is

\[
\| F \| = m \| \mathbf{a}_P \|.
\]
Using the acceleration of the particle determined previously, the magnitude of the force becomes

\[ \|F\| = m \sqrt{(\ddot{v})^2 + \left(\frac{v^2}{r}\right)^2} = m \sqrt{(\ddot{v})^2 + \left(\frac{(\dot{v} t)^2}{r^2}\right)^2}, \]

\[ = (2 \text{ kg}) \sqrt{(0.50 \text{ m/s}^2)^2 + ((0.025 \text{ m/s}^4) t^2)^2}. \]

Therefore, solving this expression for \( t \) with \( \|F\| = 16 \text{ N} \), we find that

\[ t^4 = \left(\frac{\|F\|^2 - (m \ddot{v})^2}{(m \dot{v}^2)^2}\right) r^2 = 1.02 \times 10^5 \text{ s}^4, \quad \rightarrow \quad t = 17.87 \text{ s}. \]

**Problem 4:**

A disk rotates with constant angular velocity \( \dot{\theta} = \omega = 0.25 \text{ rad/s} \). The particle \( P \) moves along a radial direction relative to the disk at constant rate \( \dot{r} = v = 1 \text{ m/s} \).

a) Find the acceleration of the particle when \( r = 0.75 \text{ m} \) in terms of radial and transverse directions (\( \hat{e}_r \) and \( \hat{e}_\theta \)).

b) What force must act on the particle at this instant to achieve this motion?

**Solution:**

a) In terms of polar coordinates, the acceleration of the particle can be written as

\[ \vec{a}_P = \left(\ddot{r} - r \dot{\theta}^2\right) \hat{e}_r + \left(r \ddot{\theta} + 2 \dot{r} \dot{\theta}\right) \hat{e}_\theta. \]

For this system, we find that

\[ \dot{r} = v, \quad \ddot{r} = 0, \quad \dot{\theta} = \omega, \quad \ddot{\theta} = 0, \]

so that the acceleration of \( P \) becomes

\[ \vec{a}_P = \left(-r \omega^2\right) \hat{e}_r + (2 v \omega) \hat{e}_\theta = \left(-\frac{3}{64} \text{ m/s}^2\right) \hat{e}_r + \left(\frac{1}{2} \text{ m/s}^2\right) \hat{e}_\theta. \]

b) Linear momentum balance implies that \( \sum \vec{F} = m \vec{a}_P \). Since the acceleration is known from above, the force required to act on the particle must be

\[ \sum \vec{F} = m \vec{a}_P = m \left[\left(-\frac{3}{64} \text{ m/s}^2\right) \hat{e}_r + \left(\frac{1}{2} \text{ m/s}^2\right) \hat{e}_\theta\right], \]

where \( m \) is the mass of the particle.
Problem 5:
The particle \( P \) moves along a spiral path, defined by:
\[
r(\theta) = \frac{1}{\pi} \theta.
\]
If the angular position of the particle is defined as \( \theta(t) = \omega t \), find the position and acceleration of \( P \) with respect to the ground after \( t = 2 \) s if \( \omega = 4 \) rad/s.

Solution:
We measure the position of the particle \( P \) with respect to the origin \( O \) in terms of polar coordinates:
\[
r_{PO} = r(t) \hat{e}_r.
\]
With this, the velocity and acceleration of \( P \) become:
\[
\vec{v}_P = \left( \dot{r}(t) \right) \hat{e}_r + \left( r(t) \dot{\theta}(t) \right) \hat{e}_\theta,
\]
\[
\vec{a}_P = \left( \ddot{r}(t) - r(t) \dddot{\theta}(t) \right) \hat{e}_r + \left( r(t) \ddot{\theta}(t) + 2 \dot{r}(t) \dot{\theta}(t) \right) \hat{e}_\theta.
\]
For the given path:
\[
\begin{align*}
\dot{r}(t) &= \frac{\omega t}{\pi}, & \dot{\theta}(t) &= \omega, \\
\ddot{r}(t) &= 0, & \ddot{\theta}(t) &= 0,
\end{align*}
\]
so that the position, velocity (not required), and acceleration at \( t = 2 \) s become:
\[
\begin{align*}
r_{PO} &= \frac{8}{\pi} \hat{e}_r, & \vec{v}_P &= \frac{4}{\pi} \hat{e}_r + \frac{32}{\pi} \hat{e}_\theta, & \vec{a}_P &= -\frac{128}{\pi} \hat{e}_r + \frac{32}{\pi} \hat{e}_\theta.
\end{align*}
\]
Finally, at this instant \( \hat{e}_r \) and \( \hat{e}_\theta \) may be related to \( \hat{i} \) and \( \hat{j} \) as:
\[
\begin{align*}
\hat{e}_r &= \cos \theta(t) \hat{e}_r + \sin \theta(t) \hat{e}_\theta, & \hat{e}_\theta &= -\sin \theta(t) \hat{e}_r + \cos \theta(t) \hat{e}_\theta, \\
&= -0.146 \hat{e}_r + 0.989 \hat{e}_\theta, & \hat{e}_\theta &= -0.989 \hat{e}_r - 0.146 \hat{e}_\theta.
\end{align*}
\]

Problem 6:
The telescoping shaft is extending at a constant rate \( \dot{r} = 0.2 \) m/s and is rotating with an angular speed given by \( \dot{\theta} = \omega(t) = 2e^{(t^2)} \) rad/s. Find the acceleration at time \( t = 10 \) s. You may leave your answer in the \( \hat{e}_r \) and \( \hat{e}_\theta \) directions.
Solution:

a) If we define \( \hat{e}_r \) to be along the shaft and \( \hat{e}_\theta \) to be in the direction of increasing \( \theta \) (as is standard), we find:

\[
\mathbf{a}_P = (\ddot{r} - r\dot{\theta}^2) \hat{e}_r + \left( r\ddot{\theta} + 2\ddot{r}\hat{e}_r + \dddot{r} \hat{e}_\theta \right).
\]

With \( \dot{r} = 0.2 \text{ m/s} \) and \( \dot{\theta} = 2e^{(t^2)} \), we see that, if \( r(0) = 0 \):

\[
\begin{align*}
\dot{r}(t) &= (0.2 \text{ m/s})t, \\
\dot{\theta} &= 2e^{(t^2 \text{ s}^{-2})} \text{ rad/s}, \\
\dddot{r} &= -5.78 \times 10^8 \text{ m/s}^2, \\
\dddot{\theta} &= 4te^{(t^2 \text{ s}^{-2})} \text{ rad/s}^3.
\end{align*}
\]

So at \( t = 10 \text{ s} \), the acceleration becomes:

\[
\mathbf{a}_P = (-5.78 \times 10^8 \text{ m/s}^2) \hat{e}_r + (2.17 \times 10^4 \text{ m/s}^2) \hat{e}_\theta.
\]

Awfully big, huh?

Problem 7:

A bug is walking outward on a record player at a constant velocity, with respect to the vinyl, of \( \mathbf{v}_B = v(\hat{i}_2 + \hat{j}_2) \). At this instant, the bug is at a position \( \mathbf{r}_{BO} = r\hat{i}_2 \). Find

a) the acceleration of the bug with respect to the vinyl;

b) the acceleration of the bug with respect to the ground.

Solution:

a) Note that the velocity of the bug with respect to the vinyl is constant. Therefore \( \mathbf{a}_B = \mathbf{0} \).

b) We simply use the acceleration formula:

\[
\mathbf{a}_B = \mathbf{a}_o + \mathbf{a}_B + \alpha_{2/3} \times \mathbf{r}_{BO} + \omega_{2/3} \times (\omega_{2/3} \times \mathbf{r}_{BO}) + 2\omega_{2/3} \times \mathbf{v}_B.
\]

and identify each term as follows:

\[
\begin{align*}
\mathbf{a}_o &= \mathbf{0}, \\
\omega_{2/3} &= \omega\hat{k}, \\
\alpha_{2/3} &= \alpha\hat{k}, \\
\mathbf{a}_B &= \mathbf{0}.
\end{align*}
\]

so that:

\[
\mathbf{a}_B = - (r\omega^2 + 2\omega v)\hat{i}_2 + (r\alpha + 2\omega v)\hat{j}_2.
\]

Note that we have expressed the acceleration of the bug with respect to the ground in terms of unit directions fixed in the disk. However, the resulting expression is nonetheless \( \mathbf{a}_B \).
Problem 8:
For the two-link system shown to the right, the base link \( CO \) of length \( \ell = 4 \) m spins with constant angular velocity \( \dot{\theta} \equiv \omega = \pi/2 \) rad/s. Meanwhile the attached link \( PQ \) of length \( 2d = 2 \) m oscillates, with the angle \( \phi \) given as

\[
\phi(t) = \phi_0 \sin(\mu t),
\]

For \( \phi_0 = \pi/6 \) rad and \( \mu = \pi \) rad/s, find

a) the velocity of \( P \), and

b) the acceleration of \( P \)

with respect to the ground at time \( t = 2 \) s.

Solution:

The basis directions in \( F_2 \) and \( F_3 \) can be related to ground as

\[
\begin{align*}
\mathbf{i}_2 &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \\
\mathbf{i}_3 &= \cos \phi \mathbf{i} + \sin \phi \mathbf{j}, \\
\mathbf{j}_2 &= -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}, \\
\mathbf{j}_3 &= -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}.
\end{align*}
\]

In general \( \theta(t) = \dot{\theta} t \), \( \dot{\phi}(t) = \mu \phi_0 \cos(\mu t) \), and \( \ddot{\phi}(t) = -\mu^2 \phi_0 \sin(\mu t) \), so that at \( t = 2 \) s,

\[
\begin{align*}
\theta(2 \text{ s}) &= \pi \text{ rad}, \\
\phi(2 \text{ s}) &= 0 \text{ rad}, \\
\dot{\phi}(2 \text{ s}) &= \pi^2/6 \text{ s}^{-1}, \\
\ddot{\phi}(2 \text{ s}) &= 0 \text{ s}^{-2}.
\end{align*}
\]

a) The velocity of \( P \) can be written as

\[
\begin{align*}
\mathbf{v}_P &= \mathbf{v}_C + \omega_{\beta_2/\mathcal{F}} \times r_{PC},
\end{align*}
\]

where \( \omega_{\beta_2/\mathcal{F}} \) is the angular velocity of link \( PQ \) and the velocity of \( C \) can be written as

\[
\begin{align*}
\mathbf{v}_C &= \mathbf{v}_O + \omega_{\beta_2/\mathcal{F}} \times r_{CO}.
\end{align*}
\]

The various components of these expressions can be expressed as

\[
\begin{align*}
\mathbf{v}_O &= \mathbf{0}, \\
\omega_{\beta_2/\mathcal{F}} &= \dot{\theta} \mathbf{k}, \\
\omega_{\beta_3/\mathcal{F}} &= \dot{\phi} \mathbf{k}, \\
r_{CO} &= \ell \mathbf{i}_2, \\
r_{PC} &= d \mathbf{i}_3.
\end{align*}
\]

As a result, the velocity of \( P \) becomes

\[
\begin{align*}
\mathbf{v}_P &= (\dot{\theta} \mathbf{k} \times \ell \mathbf{i}_2) + \dot{\theta} \mathbf{k} \times d \mathbf{i}_3, \\
&= \ell \dot{\theta} \mathbf{j}_2 + d \dot{\phi} \mathbf{j}_3.
\end{align*}
\]

Therefore at this instant \( \mathbf{j}_2 = -\mathbf{j} \) and \( \mathbf{j}_3 = \mathbf{j} \), so that

\[
\begin{align*}
\mathbf{v}_P(2 \text{ s}) &= \left(-2 \pi + \frac{\pi^2}{6}\right) \text{ m/s} \mathbf{j} = (-4.64 \text{ m/s}) \mathbf{j}.
\end{align*}
\]
b) The acceleration of $P$ can be expressed as

$$
\ddot{x}_P = \ddot{x}_C + \alpha_{\beta_3/F} \times r_{PC} + \omega_{\beta_3/F} \times (\omega_{\beta_3/F} \times r_{PC}),
$$

where $\alpha_{\beta_3/F}$ is the angular acceleration of link $PQ$ and the acceleration of $C$ is

$$
\ddot{x}_C = \ddot{x}_O + \alpha_{\beta_2/F} \times r_{CO} + \omega_{\beta_2/F} \times (\omega_{\beta_2/F} \times r_{CO}).
$$

In these equations $\ddot{x}_O = 0$, $\alpha_{\beta_2/F} = \dddot{\theta} \hat{k}$, and $\alpha_{\beta_3/F} = \dddot{\phi} \hat{k}$, so that the acceleration of $P$ becomes

$$
\ddot{x}_P = \left( \dot{\theta} \times \left( \dot{\theta} \hat{k} \times \ell \hat{i}_2 \right) \right) + \left( \dddot{\phi} \hat{k} \times d \hat{i}_3 \right) + \left( \dot{\phi} \times \left( \dddot{\phi} \hat{k} \times d \hat{i}_3 \right) \right),
$$

$$
= -\ell \dddot{\theta} \hat{i}_2 - d \dddot{\phi} \hat{i}_3 + d \dddot{\phi} \hat{j}_3.
$$

At this instant $\hat{i}_2 = -\hat{i}$, and $\hat{i}_3 = \hat{i}$, so that

$$
\ddot{x}_P(2 \text{ s}) = \left( -\frac{\pi^2}{36} + \frac{\pi^4}{36} \right) \text{m/s}^2 \hat{i} = (-7.16 \text{ m/s}^2) \hat{i}.
$$

### Problem 9:

With respect to a disk, the particle $P$ moves along a spiral with path $r_{PC} = x(t) \hat{i} + y(t) \hat{j}$, with

$$
x(t) = (\rho t) \cos(\omega t),
$$

$$
y(t) = (\rho t) \sin(\omega t),
$$

where $\rho = 0.10 \text{ m/s}$ and $\omega = 1 \text{ rad/s}$. In addition the disk is translating with constant velocity $\ddot{x}_C = v \hat{i} = 0.50 \text{ m/s}$. Find the acceleration of the particle at $t = 5 \text{ s}$.

### Solution:

We can identify an intermediate frame of reference $F_2$ fixed at the center of the disk. Thus, the acceleration of $P$ can be written as

$$
\ddot{x}_P = \ddot{x}_C + (\alpha_{\beta_2/F} \times r_{PC}) + (\omega_{\beta_2/F} \times (\omega_{\beta_2/F} \times r_{PC})) + 2 (\omega_{\beta_2/F} \times \ddot{x}_v) + \dddot{a}_P.
$$

However, the disk is not rotating and moreover the acceleration of $C$ vanishes because its velocity is constant. Therefore

$$
\ddot{x}_C = 0, \quad \alpha_{\beta_2/F} = 0, \quad \omega_{\beta_2/F} = 0,
$$

so that $\ddot{x}_P = \dddot{a}_P$. In essence $F_2$ is an inertial reference frame. We thus find that

$$
\dddot{a}_P = \dddot{x}(t) \hat{i} + \dddot{y}(t) \hat{j}.
$$
where
\[ \ddot{x}(t) = -(\rho t) \cos(\omega t) - 2 \rho \omega \sin(\omega t), \]
\[ \ddot{y}(t) = -(\rho t) \sin(\omega t) + 2 \rho \omega \cos(\omega t). \]

At \( t = 5 \) s, this reduces to
\[ \ddot{X} = (0.050 \text{ m/s}^2) \hat{i} + (0.536 \text{ m/s}^2) \hat{j}. \]

**Problem 10:**
In the mechanism shown, the inner arm (length \( \ell = 25 \text{ cm} \)) spins at a constant angular speed \( \omega_2 = 5 \text{ rad/s} \), while the disk (radius \( r = 10 \text{ cm} \)), attached to the arm at \( A \), spins at constant rate \( \omega_3 = 10 \text{ rad/s} \). At this instant, \( \theta_2 = 90^\circ \) and \( \theta_3 = 45^\circ \):

a) Determine the velocity of \( A \), the center of the disk;

b) Find the velocity and acceleration of \( P \), at the edge of the disk, with respect to the ground.

**Solution:**

a) The velocity of \( A \) can be expressed as:
\[ \ddot{X}_A = \omega_2 \hat{k} \times \ell \hat{i}_2, \]
\[ = \ell \omega_2 \hat{j}_2. \]

At this instant in time, \( \dot{\theta}_2 = -\hat{i} \), so that with the values given above:
\[ \ddot{X}_A = \left( -1.25 \text{ m/s} \right) \hat{i}. \]

b) Likewise, the velocity of \( P \) can be expressed as:
\[ \ddot{X}_P = \ddot{X}_A + \omega_2 \hat{k} \times r \hat{i}_3, \]
\[ = \ell \omega_2 \dot{j}_2 + r \omega_3 \hat{j}_3. \]

At this instant in time, \( \dot{\theta}_3 = -1/\sqrt{2} \hat{i} + 1/\sqrt{2} \hat{j} \), so that with the values given above:
\[ \ddot{X}_P = \left( -1.25 \text{ m/s} \right) \hat{i} + \left( 1 \text{ m/s} \right) \left( -\frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} \right), \]
\[ = \left( -1.96 \text{ m/s} \right) \hat{i} + \left( 0.71 \text{ m/s} \right) \hat{j}. \]
Finally, the acceleration of \( P \) is:

\[
\vec{a}_P = \vec{a}_A + \vec{\alpha}_{3/2} \times \vec{r}_{PA} + \vec{\omega}_{3/2} \times \left( \vec{\omega}_{3/2} \times \vec{r}_{PA} \right),
\]

with \( \vec{\alpha}_{3/2} = \vec{0} \), and:

\[
\vec{a}_A = \vec{\alpha}_{2/2} \times \vec{r}_{AO} + \vec{\omega}_{2/2} \times \left( \vec{\omega}_{2/2} \times \vec{r}_{AO} \right).
\]

Therefore, with \( \vec{\alpha}_{2/2} = \vec{0} \):

\[
\vec{a}_P = \vec{\omega}_{2/2} \times \left( \vec{\omega}_{2/2} \times \vec{r}_{AO} \right) + \vec{\omega}_{3/2} \times \left( \vec{\omega}_{3/2} \times \vec{r}_{PA} \right),
\]

\[
= \omega_2 \hat{k} \times \left( \omega_2 \hat{k} \times \ell \hat{i}_2 \right) + \omega_3 \hat{k} \times \left( \omega_3 \hat{k} \times r \hat{i}_3 \right),
\]

\[
= -\ell \omega_2^2 \hat{i}_2 - r \omega_3^2 \hat{i}_3,
\]

\[
= \left( -6.25 \, \text{m/s}^2 \right) \hat{i}_2 + \left( -10 \, \text{m/s}^2 \right) \hat{i}_3.
\]

At this instant:

\[
\hat{i}_2 = \hat{j}, \quad \hat{i}_3 = \frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j},
\]

so that:

\[
\vec{a}_P = \left( -7.07 \, \text{m/s}^2 \right) \hat{i} + \left( -13.32 \, \text{m/s}^2 \right) \hat{j}.
\]

---

**Problem 11:**

The body shown to the right rotates with constant angular speed \( \dot{\theta} = 5 \, \text{rad/s} \) and \( \vec{r}_{AO} = (0.50 \, \text{m}) \hat{i}_2 \). In addition, the position of the point \( P \) (relative to \( A \)) is described by \( \vec{r}_{PA} = u(t) \hat{i}_2 + v(t) \hat{j} \), with:

\[
u(t) = (0.10 \, \text{m}) \sin \left( \frac{\pi}{2} t \right), \quad v(t) = \frac{t}{8} \, \text{m/s}.
\]

If, at this instant, \( \theta = 60^\circ \) and \( t = 2 \, \text{s} \):

a) Determine \( \vec{v}_P \), the velocity of \( P \) relative to the rotating body;

b) Find the velocity and acceleration of \( P \) with respect to the ground.

---

**Solution:**

a) In terms of a \( \mathcal{F}_2 \), fixed in the rotating body, the velocity of \( P \) is found to be:

\[
\vec{v}_P = \frac{\vec{s}_2}{\vec{d}t} (\vec{r}_{PA}) = \frac{\vec{s}_2}{\vec{d}t} \left( u(t) \hat{i}_2 + v(t) \hat{j} \right),
\]

\[
= \dot{u}(t) \hat{i}_2 + \dot{v}(t) \hat{j}.
\]

Therefore, with:

\[
\dot{u}(t) = (0.05 \, \pi \, \text{m/s}) \cos \left( \frac{\pi}{2} t \right), \quad \dot{v}(t) = 0.125 \, \text{m/s},
\]
at $t = 2 \text{s}$, $\dot{x}_2 \mathbf{v}_{P}$ becomes:

\[
\dot{x}_2 \mathbf{v}_{P} = (-0.05 \, \pi \text{ m/s}) \hat{i} + (0.125 \text{ m/s}) \hat{j},
\]

\[
= (-0.157 \text{ m/s}) \hat{i} + (0.125 \text{ m/s}) \hat{j}.
\]

Similarly, we may also find $\dot{x}_2 \mathbf{a}_{P}$ to be:

\[
\dot{x}_2 \mathbf{a}_{P} = \dot{x}_2 \frac{d^2}{dt^2} (\mathbf{r}_{P,A}) = \dot{x}_2 \frac{d^2}{dt^2} \left(u(t) \hat{i} + v(t) \hat{j}\right),
\]

\[
= \ddot{u}(t) \hat{i} + \ddot{v}(t) \hat{j}.
\]

and at this instant, $\dot{x}_2 \mathbf{a}_{P} = \mathbf{0}$.

b) Since the point $O$ is fixed in the body (as well as the ground), the velocity and acceleration of $P$ with respect to the ground may, in general, be written as:

\[
\dot{x}_2 \mathbf{v}_{P} = \dot{x}_2 \mathbf{v}_{O} + \dot{x}_2 \mathbf{v}_{P} + \omega_{2/F} \times \mathbf{r}_{P,O},
\]

\[
\dot{x}_2 \mathbf{a}_{P} = \dot{x}_2 \mathbf{a}_{O} + \dot{x}_2 \mathbf{a}_{P} + \omega_{2/F} \times \mathbf{r}_{P,O} + \omega_{2/F} \times \left(\omega_{2/F} \times \mathbf{r}_{P,O}\right) + 2 \omega_{2/F} \times \dot{x}_2 \mathbf{v}_{P}.
\]

For this system, we find that:

\[
\dot{x}_2 \mathbf{v}_{O} = \mathbf{0}, \quad \dot{x}_2 \mathbf{a}_{O} = \mathbf{0}, \quad \omega_{2/F} = \omega = (5 \, \text{rad/s}) \hat{k}, \quad \alpha_{2/F} = \mathbf{0}.
\]

In addition, at this instant:

\[
\dot{x}_2 \mathbf{a}_{P} = \mathbf{0}, \quad \mathbf{r}_{P,O} = (0.50 \text{ m}) \hat{i} + (0.25 \text{ m}) \hat{j}.
\]

Finally, the velocity and acceleration of $P$ become:

\[
\dot{x}_2 \mathbf{v}_{P} = \dot{x}_2 \mathbf{v}_{O} + \dot{x}_2 \mathbf{v}_{P} + \omega_{2/F} \times \mathbf{r}_{P,O},
\]

\[
= \mathbf{0} + ((-0.157 \text{ m/s}) \hat{i} + (0.125 \text{ m/s}) \hat{j})
\]

\[
+ \left((5 \text{ rad/s}) \hat{k} \times ((0.50 \text{ m}) \hat{i} + (0.25 \text{ m}) \hat{j})\right),
\]

\[
= (-1.407 \text{ m/s}) \hat{i} + (2.625 \text{ m/s}) \hat{j}.
\]

\[
\dot{x}_2 \mathbf{a}_{P} = \dot{x}_2 \mathbf{a}_{O} + \dot{x}_2 \mathbf{a}_{P} + \omega_{2/F} \times \mathbf{r}_{P,O} + \omega_{2/F} \times \left(\omega_{2/F} \times \mathbf{r}_{P,O}\right) + 2 \omega_{2/F} \times \dot{x}_2 \mathbf{v}_{P},
\]

\[
= \mathbf{0} + \mathbf{0} + \left((5 \text{ rad/s}) \hat{k} \times ((0.50 \text{ m}) \hat{i} + (0.25 \text{ m}) \hat{j})\right)
\]

\[
+ 2 \left((5 \text{ rad/s}) \hat{k} \times ((-0.157 \text{ m/s}) \hat{i} + (0.125 \text{ m/s}) \hat{j})\right)
\]

\[
= (-13.75 \text{ m/s}^2) \hat{i} + (-7.82 \text{ m/s}^2) \hat{j}.
\]

Note that the velocity and acceleration of $P$ could be expressed through the intermediate point $A$, instead of $O$. In either case the final result is the same.

**Problem 12:**

The rod $AB$ is confined to move along the inclined planes as shown. If the acceleration of point $A$ has a magnitude of $5.0 \text{ m/s}^2$ and a speed $2.0 \text{ m/s}$, both directed upward when the bar is horizontal, find the angular acceleration of the bar at this instant.
Problem 13:
At time $t = 0$ the disk of mass $m$ and radius $r$ has an angular velocity of $\omega \hat{k}$. If a block of mass $m$ is attached to the cord, find:

a) the angular acceleration of the disk at $t = 0^+$;

b) the distance the block falls after $t$ seconds.

---

2 Particle Dynamics

Problem 14:
A particle of mass $m$ moves along a frictionless circular hoop of radius $r$, and is subject to gravity and a transverse force $F \hat{e}_\theta$.

a) Find the angular acceleration $\ddot{\theta}$ of the particle as a function of $\theta$;

b) If the particle is released from rest at $\theta = 0$, determine its speed when $\theta = 90^\circ$ if $F = 2mg$.

---

Solution:

a) Using the coordinate $\theta$ as provided, the acceleration of the particle can be written as

$$\alpha = (-r \dot{\theta}^2) \hat{e}_r + (r \ddot{\theta}) \hat{e}_\theta.$$  

In addition, a free body diagram for the particle can be constructed as shown to the right. The force $N \hat{e}_r$ represents the normal force acting between the hoop and the particle while the gravitational force is in the $\hat{j}$ direction. Finally, the $\hat{i}$ and $\hat{j}$ directions can be written in terms of $\hat{e}_r$ and $\hat{e}_\theta$ as

$$\hat{i} = \cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta,$$

$$\hat{j} = \sin \theta \hat{e}_r + \cos \theta \hat{e}_\theta.$$  

Applying linear momentum balance to this particle, the equations of motion can be written
as

\[ \sum F = m \vec{a}_G \]

\[ (N - mg \sin \theta) \dot{e}_r + (F - mg \cos \theta) \dot{e}_\theta = \left( -m r \dot{\theta}^2 \right) \dot{e}_r + \left( m r \ddot{\theta} \right) \dot{e}_\theta. \]

Therefore, the component of this equation in the \( \dot{e}_\theta \) direction may be solved for \( \ddot{\theta} \) to yield

\[ \ddot{\theta} = \frac{F - mg \cos \theta}{mr}. \]

b) In the above result, the second derivative of \( \theta \) with respect to time is a function of \( \theta \) alone. Thus we can relate \( \dot{\theta} \) and \( \theta \) as

\[ \int_0^t \{ \ddot{\theta} = \frac{F - mg \cos \theta}{mr} \} \dot{\theta} \, dt, \]

\[ \int_0^t \dot{\theta} \, dt = \int_0^t \left( \frac{F - mg \cos \theta}{mr} \right) \dot{\theta} \, dt, \]

\[ \frac{\dot{\theta}^2(t) - \dot{\theta}^2(0)}{2} = \frac{F}{mr} \left( \theta(t) - \theta(0) \right) - \frac{g}{r} \left( \sin \theta(t) - \sin \theta(0) \right). \]

When the particle is released from rest at \( \theta = 0, \dot{\theta}(0) = 0 \) and \( \theta(0) = 0 \), so that this reduces to

\[ \frac{\dot{\theta}^2(t)}{2} = \frac{F}{mr} \theta(t) - \frac{g}{r} \sin \theta(t). \]

With \( F = 2mg \), when \( \theta(t_f) = \pi/2 \), the value of \( \dot{\theta}(t_f) \) becomes

\[ \dot{\theta}(t_f) = \sqrt{(2 \pi - 1) \frac{g}{r}}. \]

Finally, the speed of the particle is \( \dot{s} = r \dot{\theta} \), and at this instant

\[ \dot{s}(t_f) = r \dot{\theta}(t_f) = \sqrt{(2 \pi - 1) g r}. \]

**Problem 15:**
The upper block rests on a rough surface with coefficient of friction \( \mu \) and is subject to a force \( F = -F_0 \hat{i} \) with \( F_0 > 0 \).

a) Find the acceleration of the lower block assuming the upper block slides to the right.

b) If the system is released from rest, for what values of the coefficient of friction \( \mu \) will the system remain stationary?
Solution:

a) We begin by defining two coordinates, $x$ and $y$, as shown in the figure, which measure the displacement of the two masses. As the upper block moves to the right by some distance $x$, the pulley moves down by half the distance, so that $y = x/2$. Finally, the acceleration of each block is

\[ \vec{a}_{c_1} = \ddot{x} \hat{i}, \quad \vec{a}_{c_2} = -\ddot{y} \hat{j}. \]

An appropriate free body diagram for each mass is shown to the right, as well as one for the massless pulley. Notice that the forces acting on the pulley must equilibrate because this element is massless. Also, the unknown tension in the cable is $T$, while the friction force acting on the upper block is $f_r \hat{i}$. When the block is sliding the magnitude of this force is related to the velocity of the upper block as

\[ f_r = -\mu N \sgn(\dot{x}), \]

so that if the block is sliding to the right, $\dot{x} > 0$, $f_r < 0$, and the friction force is orientated in the $-\hat{i}$ direction.

Applying linear momentum balance to each block yields the following equations of motion

\[ \sum \vec{F} = (f_r + T - F_0) \hat{i} + (N - m g) \hat{j} = m \ddot{x} \hat{i} = m \vec{a}_{c_1}, \]
\[ \sum \vec{F} = (2 T - m g) \hat{j} = -m \ddot{y} \hat{j} = m \vec{a}_{c_2}. \]

Therefore $N = m g$ and, substituting in for the friction force (with $\dot{x} > 0$), the remaining equations reduce to

\[ -\mu m g + T - F_0 = m \ddot{x}, \quad 2 T - m g = -m \ddot{y}. \]

Eliminating the unknown tension $T$ from these equations leads to

\[ \frac{m g}{2} - \mu m g - F_0 = m \left( \ddot{x} + \frac{\ddot{y}}{2} \right). \]

Finally, using the kinematic relation between $x$ and $y$, we may solve for $\ddot{y}$ as

\[ \ddot{y} = \frac{g}{2} (1 - 2 \mu) - \frac{F_0}{m}. \]

b) If the block is stationary, the acceleration of each block vanishes, so that $\ddot{x} = \ddot{y} = 0$, and the equations of motion become

\[ \sum \vec{F} = (f_r + T - F_0) \hat{i} + (N - m g) \hat{j} = 0, \]
\[ \sum \vec{F} = (2 T - m g) \hat{j} = 0. \]
Eliminating the unknown tension, these equations of motion reduce to
\[ f_r + \frac{m g}{2} - F_0 = 0, \quad N - m g = 0, \]
and solving for the friction force and the normal load
\[ f_r = F_0 - \frac{m g}{2}, \quad N = m g. \]
Therefore the stationary state is valid provided \(|f_r| \leq \theta N\), or
\[ \theta \geq \frac{|F_0| \cos \phi - 1/2}{m g}. \]

**Problem 16:**

A pair of blocks, each of mass \( m = 2 \) kg is being pulled up a smooth surface inclined at an angle \( \phi = 30^\circ \) by a force \( \mathbf{F} = F_0 \hat{e}_1 \), with \( F = 10 \) N.

a) Find the acceleration of the blocks;

b) What is the tension in the cable connecting the two masses?

**Solution:**

a) We begin by defining the directions \( \hat{e}_1 \) and \( \hat{e}_2 \) normal and tangential to the surface, while the coordinate \( x \) measures the displacement of the blocks up the plane, in the \( \hat{e}_1 \) direction. In addition, the directions \( \hat{i} \) and \( \hat{j} \) can be expressed in the \( \hat{e}_1 \) and \( \hat{e}_2 \) directions as

\[
\hat{i} = \cos \phi \hat{e}_1 - \sin \phi \hat{e}_2, \\
\hat{j} = \sin \phi \hat{e}_1 + \cos \phi \hat{e}_2.
\]

A free body diagram is shown to the right for each mass. Note that the acceleration of each block is identical, so that \( \hat{x} a_{G_1} = \hat{x} a_{G_2} = \hat{x} \dot{\hat{e}}_1 \). Applying linear momentum balance to each mass leads to

\[
(F_0 - T) \hat{e}_1 + N_1 \hat{e}_2 - m g \hat{j} = m \ddot{x} \hat{e}_1, \\
T \hat{e}_1 + N_2 \hat{e}_2 - m g \hat{j} = m \ddot{x} \hat{e}_1.
\]

If these equations are expressed only in the \( \hat{e}_1 \) and \( \hat{e}_2 \) directions

\[
(F_0 - T - m g \sin \phi) \hat{e}_1 + (N_1 - m g \cos \phi) \hat{e}_2 = m \ddot{x} \hat{e}_1, \\
(T - m g \sin \phi) \hat{e}_1 + (N_2 - m g \cos \phi) \hat{e}_2 = m \ddot{x} \hat{e}_1,
\]
Eliminating $T$ from these equations of motion in the $\hat{e}_1$ direction yields

$$2m \ddot{x} = F_0 - 2m g \sin \phi,$$

so that solving for the acceleration

$$\ddot{x} = \frac{F_0}{2m} - g \sin \phi \quad \Rightarrow \quad \ddot{x} = 1.25 \text{ m/s}^2.$$

b) Likewise, from the equations of motion the tension $T$ can be determined to be

$$T = \frac{F_0}{2} \quad \Rightarrow \quad T = 5 \text{ N}.$$

Problem 17:
A projectile is launched with initial speed $v$ at an angle of inclination of $\phi = 30^\circ$. Find the range of initial speeds so that the particle strikes the wall of height $h = 3 \text{ m}$ that is located a distance $d = 25 \text{ m}$ down the surface that is inclined $\psi = 15^\circ$.

Solution:
If the position of the projectile is measured as $\mathbf{r}_{PO} = x(t) \hat{i} + y(t) \hat{j}$, then its velocity is $\dot{\mathbf{v}}_p = \dot{x} \hat{i} + \dot{y} \hat{j}$, so that

$$\dot{x}(0) = v_0 \cos \phi, \quad \dot{y}(0) = v_0 \sin \phi,$$

where $v_0$ is the initial speed of the particle. Further, the acceleration is $\dot{\mathbf{a}}_p = \ddot{x} \hat{i} + \ddot{y} \hat{j}$ and since the only force that acts on the particle is due to gravity, the velocity and position of the particle may be determined to be

$$\ddot{x}(t) = 0, \quad \ddot{y}(t) = -g,$$

$$\dot{x}(t) = (v_0 \cos \phi), \quad \dot{y}(t) = -g t + (v_0 \sin \phi),$$

$$x(t) = (v_0 \cos \phi) t, \quad y(t) = -\frac{g t^2}{2} + (v_0 \sin \phi) t.$$

Therefore, the displacement in the $\hat{j}$ direction may be expressed in terms of $x$ as

$$y(x) = -\frac{g}{2} \left( \frac{x}{v_0 \cos \phi} \right)^2 \frac{\sin \phi}{\cos \phi} x.$$

Solving for $v_0$, the initial speed necessary for the particle to pass through a position $\mathbf{r}_{PO} = x \hat{i} + y \hat{j}$ yields

$$v_0 = x \sqrt{\frac{g}{2 \cos \phi (x \sin \phi - y \cos \phi)}}.$$

The final position of the particle as it hits the wall is

$$\mathbf{r}_{PO}(t_f) = d \hat{e}_1 + z \hat{j} = (d \cos \psi) \hat{i} + (z - d \sin \psi) \hat{j},$$

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where at the base of the wall \( z = 0 \) and at the top of the wall \( z = h \). Therefore the initial speed required to hit the wall at height \( z \) can be written as

\[
v_0 = (d \cos \psi) \sqrt{\frac{g}{2 \cos \phi (d \sin(\phi + \psi) - z \cos \phi)}}.
\]

Finally, substituting numerical values into this expression, the minimum velocity required to hit the wall occurs for \( z = 0 \), and is \( v_{0,\text{min}} = 13.67 \text{ m/s} \). Likewise, the maximum velocity required to hit the wall occurs for \( z = h = 3 \text{ m} \), and is \( v_{0,\text{max}} = 14.80 \text{ m/s} \). Thus the range of initial speeds required to strike the wall is

\[
13.67 \text{ m/s} = v_{0,\text{min}} \leq v_0 \leq v_{0,\text{max}} = 14.80 \text{ m/s}.
\]

**Problem 18:**

As illustrated in the figure, the projectile, subject to only the influence of gravity, is launched with initial speed \( v_0 \) at an angle \( \theta_0 \). For what initial inclinations will the projectile pass through the opening of width \( d \) at a minimum height \( h \) and a distance \( \ell \) from the initial location? Assume that \( g = 10 \text{ m/s}^2 \) and

\[
\ell = 10 \text{ m}, \quad h = 3 \text{ m}, \quad d = 0.25 \text{ m}, \quad v_0 = 3.81 \text{ m/s}.
\]

**Problem 19:**

A block of mass \( m_A \) is initially at rest on a rough surface with coefficient of friction \( \mu \). A particle of mass \( m_B \) is moving with velocity \( v_B \hat{i} \), strikes the block and is embedded, so that the coefficient of restitution is \( e = 0 \). Find the distance the combined block and particle slide after the impact before coming to rest.
Problem 20:
A pendulum with of mass $m$ and length $\ell$ is released horizontally from rest. As this mass swings through an angle $\phi$ it impacts a massless spring with stiffness $k$.

a) What is the maximum compression of the spring?

b) If the mass remains attached to the spring, what is the amplitude of the resulting oscillations?

Solution:

a) As we are interested in the state of the system at some final time given the initial state, we formulate the problem in terms of work and energy. Moreover, since all external forces are conservative we use conservation of energy.

We measure the compression of the spring as $\delta$ as shown in the figure above. Therefore the potential energy of the system can be written as

$$V = mgh + \frac{1}{2}k\delta^2,$$

where $h$ is the height of the mass above the ground. In this problem the system is released from rest and at the system is again at rest at the maximum compression of the spring. Therefore both the initial and final kinetic energy is zero.
The initial height and spring displacement are

\[ h_i = 0, \quad \delta_i = 0, \]

while the final height and displacement are

\[ h_f = -\ell \sin \phi - \delta_{\text{max}} \cos \phi, \quad \delta_f = \delta_{\text{max}}. \]

Therefore conservation of energy yields

\[ T_i + V_i = T_f + V_f, \]

\[ 0 = -mg (\ell \sin \phi + \delta_{\text{max}} \cos \phi) + \frac{1}{2} k \delta_{\text{max}}^2. \]

Finally, solving for \( \delta_{\text{max}} \) yields two solutions

\[ \delta_{\text{max}} = \frac{mg \cos \phi}{k} \pm \sqrt{\left( \frac{mg \cos \phi}{k} \right)^2 - \frac{2mg \ell \sin \phi}{k}}. \]

Therefore the maximum compression of the spring is the largest of these two, or

\[ \delta_{\text{max}} = \frac{mg \cos \phi}{k} + \sqrt{\left( \frac{mg \cos \phi}{k} \right)^2 - \frac{2mg \ell \sin \phi}{k}}. \]

b) The second solution to the above quadratic equation physically corresponds to the extension of the spring during the rebound, assuming the mass remains attached to the spring. Thus the amplitude \( A \) of the resulting oscillations is half the distance between these two extrema, or

\[ A = \sqrt{\left( \frac{mg \cos \phi}{k} \right)^2 - \frac{2mg \ell \sin \phi}{k}}. \]

**Problem 21:**
The point mass \( m \) shown to the right is supported on the rotating bar by a spring with unstretched length \( \ell \) and stiffness \( k \). If the bar is rotating at constant angular speed \( \dot{\theta} = \omega \):

a) find the acceleration of the mass relative to the bar (in the radial direction) as a function of \( r(t) \), the radial displacement;

b) when the mass is in equilibrium relative to the bar (in the radial direction), what is the steady-state compression of the spring?

**Solution:**

a) In addition to the coordinates \( r(t) \) and \( \theta(t) \) as shown in the figure, we also identify the stretch in the spring as \( \delta(t) \), related to \( z(t) \) as

\[ \delta(t) = d - \ell - r(t). \]
The directions $\hat{e}_r$ and $\hat{e}_\theta$ are radial and tangent to the rotating bar.

Neglecting gravity, a free-body diagram for this system is shown to the right. The force $N\hat{e}_\theta$ represents the force from the bar applied to the point mass while the force from the spring is $k\delta(t)\hat{e}_r$.

Finally, in terms of the coordinates $r(t)$ and $\theta(t)$, the acceleration of the mass can be written as

$$\mathbf{a}_c = (\ddot{r}(t) - r(t)\omega^2)\hat{e}_r + (2\dot{r}(t)\omega)\hat{e}_\theta,$$

where we have made use of the constant angular velocity of the bar, so that $\dot{\theta}(t) = 0$.

Applying momentum balance to this system

$$\sum \mathbf{F} = (k\delta(t))\hat{e}_r + (N(t))\hat{e}_\theta = m \left((\ddot{r}(t) - r(t)\omega^2)\hat{e}_r + (2\dot{r}(t)\omega)\hat{e}_\theta\right) = m\mathbf{a}_c.$$

Thus, taking the component in the $\hat{e}_r$ direction

$$k\delta(t) = m(\ddot{r}(t) - r(t)\omega^2).$$

Finally, solving for the acceleration along the bar, we obtain

$$\ddot{r}(t) = \frac{k}{m} \left(\frac{(d - \ell - r(t)) + r(t)\omega^2}{m} - \frac{(k - m\omega^2)}{m} r(t)\right).$$

b) The equilibrium position of the mass relative to the bar implies that $\ddot{r}(t) \equiv 0$, so that the above equation can be solved to yield

$$r_{eq} = \frac{k(d - \ell)}{k - m\omega^2}.$$  

---

Problem 22:
The two blocks shown to the right, each of mass $m = 2 \text{ kg}$, rest on a rough surface with coefficient of friction $\mu = 0.25$. The initial velocity of the left block is $v\hat{i}$, with $v = 1.50 \text{ m/s}$ while the second block, located a distance $d = 0.25 \text{ m}$ away is initially at rest. If the coefficient of restitution between the blocks is $e = 0.80$, find the distance the right block slides after the impact before coming to rest.

Solution:
The motion of this system can be divided into three phases. In the first, the block $A$ slides to the right. The second phase describes the impact between blocks $A$ and $B$, while the final phase describes the motion of block $B$ as it comes to rest. The first and third phase can be described using the work-energy formulation while the collision is best described using an impulse-momentum formulation.
Because the blocks are identical, a free-body diagram of either block is shown to the right. Since the motion is only in the $\hat{i}$ direction, only the force in the $\hat{i}$ direction contributes to the work. The work-energy formulation for either sliding block yields

$$\int_0^{x(t)} (-\mu m g \hat{i}) \cdot (dx \hat{i}) = \frac{m}{2} (\dot{x}^2(t) - \dot{x}^2(0)),$$

so that the velocity of block $A$ at the impact, $\dot{x}_{A,1}$ is determined from

$$-\mu m g d = \frac{m}{2} (\dot{x}_{A,1}^2 - v^2).$$

Solving for the velocity of block $A$ just before the impact, $\dot{x}_{A,1}$

$$\dot{x}_{A,1} = \sqrt{v^2 - 2 \mu g d}.$$

Across the impact conservation of momentum holds

$$m_A \dot{x}_{A,1} + m_B \dot{x}_{B,1} = m_A \dot{x}_{A,2} + m_B \dot{x}_{B,2}, \quad \rightarrow \dot{x}_{A,1} = \dot{x}_{A,2} + \dot{x}_{B,2}.$$

Together with the definition of the coefficient of restitution

$$e = \frac{\dot{x}_{B,2} - \dot{x}_{A,2}}{\dot{x}_{A,1} - \dot{x}_{B,1}}, \quad \rightarrow e \dot{x}_{A,1} = \dot{x}_{B,2} - \dot{x}_{A,2}.$$

Therefore, solving for the velocity of block $B$ after the collision, $\dot{x}_{B,2}$

$$\dot{x}_{B,2} = \frac{1}{2} e \dot{x}_{A,1}.$$

Over the final interval of the motion we again use a work-energy formulation for block $B$. If the distance traveled by block $B$ before coming to rest is $\delta$, then $\dot{x}_{B,3} = 0$ and using the relation obtained above

$$-\mu m g \delta = -\frac{m}{2} \dot{x}_{B,2}^2,$$

and solving for $\delta$

$$\delta = \frac{1}{2 \mu g} \dot{x}_{B,2}^2,$$

$$= \frac{(1 + e)^2}{8 \mu g} \dot{x}_{A,1}^2,$$

$$= \frac{(1 + e)^2}{8 \mu g} (v^2 - 2 \mu g d) = \frac{(1 + e)^2}{4} \left( \frac{v^2}{2 \mu g} - d \right).$$

Finally, with the supplied values, we find that block $B$ slides a distance $\delta = 0.17$ m before coming to rest.
Problem 23:
In the system shown to the right, the two masses ($m = 1 \text{ kg}$) are connected by a cable wrapped around a massless pulley. If the system is released from rest, with the spring ($k = 250 \text{ N/m}$) compressed by a distance $d = 0.15 \text{ m}$, from its unstretched position:

a) find the velocity of the mass on the right when the spring is unstretched;

b) what is the maximum velocity of the mass on the left?

Solution:
We choose a work-energy formulation to describe the motion of this system. Specifically, all forces are conservative so that the total energy of the system is conserved

$$T_i + V_i = T_f + V_f,$$

where $T$ and $V$ are the kinetic and potential energies of the system.

The displacement of each mass relative to its position when the spring is unstretched is described by $x$, so that the kinetic and potential energies can be written as

$$T = \frac{1}{2} (m) \dot{x}^2 + \frac{1}{2} (4m) \dot{x}_2 = \frac{1}{2} (5m) \dot{x}^2,$$

$$V = \frac{1}{2} k x^2 + m g x - (4m) g x = \frac{1}{2} k x^2 - (3m) g x.$$

Notice that the potential energy has been taken relative to the configuration of the system when the spring is unstretched.

For an initial compression in the spring of $d$, so that $x_i = -d$, conservation of energy yields

$$T_i + V_i = T_f + V_f,$$

$$\frac{k}{2} d^2 + 3 m g d = \frac{5m}{2} \ddot{x}^2 + \frac{k}{2} x^2 - 3 m g x.$$

a) When the spring is unstretched, $x_f = 0$ and solving the above for $\ddot{x}_f$ yields

$$\ddot{x}_f = \sqrt{\frac{k d^2 + 6 m g d}{5m}}.$$

Using the provided parameter values, $\ddot{x}_f = 1.70 \text{ m/s}$. 

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b) The maximum kinetic energy of the system occurs when the potential energy is minimized. Thus finding the location \( x = x_m \) at which this occurs, \( V_{\text{min}} \)
\[
\frac{dV}{dx} = k x_m - 3 m g = 0, \quad \rightarrow x_m = \frac{3 m g}{k}.
\]

Therefore, at this position the velocity of the masses can be determined to be
\[
\dot{x}_m = \sqrt{\left(\frac{k d}{5 k m}\right)^2 + 6 \left(\frac{m g}{k d}\right) G \left(\frac{9 m g}{k d}\right) + \frac{9 (m g)^2}{5 k m}}.
\]

Using the numerical values for this problem, \( \dot{x}_m = 1.89 \, \text{m/s} \).

**Problem 24:**
A particle of mass \( m \) moves along a frictionless circular bar of radius \( r \), and is subject to gravity and a horizontal force \( \dot{F} \hat{i} \).

a) What is the angular acceleration \( \ddot{\theta} \) of the particle?

b) Find the magnitude of the force that the bar applies to the particle.

Note that your answers may (or may not) depend on \( \theta \) and \( \dot{\theta} \).

**Solution:**

a) With the coordinate \( \theta \) as defined above, the acceleration of the mass \( P \) can be written as
\[
x a_r = \left( -r \dot{\theta}^2 \right) \hat{e}_r + \left( r \ddot{\theta} \right) \hat{e}_\theta.
\]

A free-body diagram for the mass is shown to the right. In addition to the gravitational force and the applied force, a normal force in the \( \hat{e}_r \) direction represents the interaction between the frictionless hoop and the mass. Therefore the equations of motion can be written as
\[
\sum F = m x a_r, \\
F \hat{i} - m g \hat{j} + N \hat{e}_r = m \left( \left( -r \dot{\theta}^2 \right) \hat{e}_r + \left( r \ddot{\theta} \right) \hat{e}_\theta \right).
\]

Relating the \((\hat{i}, \hat{j})\) directions to the \((\hat{e}_r, \hat{e}_\theta)\) directions yields
\[
\hat{i} = \cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta, \quad \hat{j} = \sin \theta \hat{e}_r + \cos \theta \hat{e}_\theta,
\]
Therefore, to determine the angular acceleration of the particle, we take the components of the this equation in the \( \hat{e}_\theta \) direction, which eliminates the unknown normal force \( N \). This becomes

\[-F \sin \theta - m g \cos \theta = m r \ddot{\theta}.\]

Finally, solving for \( \ddot{\theta} \)

\[\ddot{\theta} = - \frac{F \sin \theta + m g \cos \theta}{m r}.\]

b) The force between the bar and the particle, described as \( N \text{univ}_r \), can be determined from the equations of motion in the \( \hat{e}_r \) direction as

\[F \cos \theta - m g \sin \theta + N = -m r \dot{\theta}^2,\]

so that solving for \( N \) yields

\[N = m g \sin \theta - F \cos \theta - m r \dot{\theta}^2.\]

Problem 25:
The two blocks of mass \( m \) are connected by a cable. The left surface is rough with a coefficient of friction \( \mu = 0.25 \) while the right surface is smooth (\( \mu_{\text{right}} = 0 \)). The left surface is inclined at an angle \( \phi \) and the angle between the surfaces is \( 90^\circ \).

a) for what range of the angle \( \phi \) do the blocks slip?

b) assuming the blocks do slip, what are their accelerations?

Solution:
As shown in the figure above, we measure the displacement of each block by the coordinate \( x(t) \) and identify the directions \( (\hat{i}_2, \hat{j}_2) \) aligned with the surface. Therefore the acceleration of the block resting on the rough surface is \( ^r \mathbf{a}_{G_L} = \ddot{x} \hat{i}_2 \), while the acceleration of the block on the smooth surface is \( ^r \mathbf{a}_{G_R} = -\ddot{x} \hat{j}_2. \)
Free-body diagrams for each block are shown to the right. Notice that a force due to friction acts only the block resting on the rough surface, although both blocks are subject to forces normal to their respective surfaces.

On the left block the equations of motion can therefore be written as

\[ \sum F = m a, \]

\[ T \hat{i} + f \hat{i} + N \hat{j} - mg \hat{j} = m(\ddot{x} \hat{i}), \]

while on the right block

\[ \sum F = m a, \]

\[ T \hat{j} + N \hat{i} - mg \hat{j} = m(-\ddot{x} \hat{j}). \]

The direction of the gravitational force \( \hat{j} \) can be written in terms of \((\hat{i}, \hat{j})\) as

\[ \hat{j} = \sin \phi \hat{i} + \cos \phi \hat{j}. \]

Finally, taking components of these equations in terms of \((\hat{i}, \hat{j})\), we obtain the four scalar equations

\[ T + f - mg \sin \phi = m \ddot{x}, \quad N - mg \cos \phi = 0, \]

\[ N - mg \sin \phi = 0, \quad T - mg \cos \phi = -m \ddot{x}. \]

We can immediately solve for the unknown normal forces acting on the two blocks as

\[ N_l = mg \cos \theta, \quad N_r = mg \sin \theta. \]

Eliminating the unknown tension yields

\[ 2m \ddot{x} = mg \cos \phi - \sin \phi + f, \]

where \( f \) is the (as yet) unknown friction force acting on the right block. If the blocks slip \( \dot{x} \neq 0 \) and

\[ f_{\text{slip}} = -\mu N \text{sgn}(\dot{x}) = -\mu mg \cos \phi \text{sgn}(\dot{x}), \]

while if the blocks are stationary then \( \dot{x} = 0 \) and

\[ |f| \leq \mu N \leq \mu mg \cos \phi. \]

a) The blocks slip provided the friction force \( f \) is unable to maintain static equilibrium, which implies that for equilibrium

\[ |f_{\text{eq}}| = |mg (\cos \phi - \sin \phi)| \leq \mu mg \cos \phi, \]

or

\[ -\mu mg \cos \phi \leq mg (\cos \phi - \sin \phi) \leq \mu mg \cos \phi. \]
This can be solved for \( \phi \) so that static equilibrium requires

\[ 1 - \mu \leq \tan \phi \leq 1 + \mu. \]

Therefore, the system slides for \( \phi \) outside this interval, which for these parameter values yields

\[ |\phi| < 36.9^\circ, \quad \text{and} \quad 51.3^\circ < |\phi| < 90^\circ. \]

b) If the blocks slip then the friction force is

\[ f_t = -\mu m g \cos \phi \, \text{sgn}(\dot{x}), \]

so that the \( \ddot{x} \) may be determined to be

\[ \ddot{x} = \frac{g}{2} ((1 - \mu \, \text{sgn}(\dot{x})) \cos \phi - \sin \phi). \]

Problem 26:
As illustrated in the figure, the projectile, subject to only the influence of gravity, is launched with initial speed \( v_0 \), an angle \( \theta_0 = 60^\circ \), and a distance \( \ell = 10 \) m from the edge. For what range of initial speeds will the mass will reach the lower platform?

Solution:

a) As shown in the figure, the position of the projectile \( P \) with respect to the firing point \( O \) is

\[ \mathbf{r}_{PO} = x(t) \mathbf{i} + y(t) \mathbf{j}, \]

so that \( x(t) \) measures the horizontal distance travelled while \( y(t) \) measures the vertical displacement of \( P \). The initial position of the projectile is

\[ \mathbf{r}_{PO}(0) = x(0) \mathbf{i} + y(0) \mathbf{j} = 0, \]

while the initial velocity is

\[ \mathbf{v}_P(0) = \dot{x}(0) \mathbf{i} + \dot{y}(0) \mathbf{j} = v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j}. \]

The acceleration of the projectile is then determined to be

\[ \mathbf{a}_P = \ddot{x} \mathbf{i} + \ddot{y} \mathbf{j}. \]

A suitable free-body diagram for the projectile is shown to the right, where only the force due to gravity acts on \( P \). Therefore the equations of motion can be written as

\[ \sum \mathbf{F} = m \mathbf{a}_P, \]

\[ -m g \mathbf{j} = m (\ddot{x} \mathbf{i} + \ddot{y} \mathbf{j}). \]

or taking components in the \( \mathbf{i} \) and \( \mathbf{j} \) directions

\[ \ddot{x} = 0, \quad \ddot{y} = -g. \]
Using the initial conditions
\[ x(0) = 0, \quad \dot{x}(0) = v_0 \cos \theta, \]
\[ y(0) = 0, \quad \dot{y}(0) = v_0 \sin \theta, \]
these equations may be solved to yield
\[ x(t) = (v_0 \cos \theta) t, \]
\[ y(t) = -\frac{g}{2} t^2 + (v_0 \sin \theta) t. \]

Notice that we can eliminate \( t \) from these two equations to directly relate \( x \) and \( y \) as
\[ y = -\frac{g}{2(v_0 \cos \theta)^2} x^2 + \frac{\sin \theta}{\cos \theta} x = \frac{(2 v_0^2 \sin \theta \cos \theta - g x) x}{2 v_0^2 \cos^2 \theta}. \]

At the minimum initial speed \( v_0 = v_{\text{min}} \) the projectile just clears upper edge at \((x, y) = (\ell, 0)\), so that
\[ 0 = \frac{(2 v_{\text{min}}^2 \sin \theta \cos \theta - g \ell) \ell}{2 v_{\text{min}}^2 \cos^2 \theta}. \]
Solving for \( v_{\text{min}} \)
\[ v_{\text{min}} = \sqrt{\frac{g \ell}{2 \sin \theta \cos \theta}} = \sqrt{\frac{g \ell}{\sin 2 \theta}}. \]

Likewise, at the maximum initial speed the projectile reaches the lower edge, which is located at \((x, y) = (3\ell/2, -\ell/2)\), so that
\[ -\frac{\ell}{2} = \frac{(2 v_{\text{max}}^2 \sin \theta \cos \theta - \frac{3}{2} g \ell) \frac{3 \ell}{2}}{2 v_{\text{max}}^2 \cos^2 \theta}, \]
and solving for \( v_{\text{max}} \)
\[ v_{\text{max}} = \sqrt{\frac{9 g \ell}{4 (3 \cos \theta \sin \theta + \cos^2 \theta)}} = \sqrt{\frac{9 g \ell}{2 (1 + \cos 2 \theta + 3 \sin 2 \theta)}}. \]

Finally, the mass reaches the lower platform for \( v_{\text{min}} < v_0 < v_{\text{max}} \), or, with the values given above
\[ 10.64 \, \text{m/s} < v_0 < 11.94 \, \text{m/s}. \]

**Problem 27:**
As illustrated in the figure, the projectile, subject to only the influence of gravity, is launched with initial speed \( v_0 \) at an angle \( \theta_0 \). The surface is inclined at an angle \( \phi \). For what initial velocity \( v_0 \) will the range (up the surface) of the projectile be \( d \)?
Problem 28:
Blocks $A$ and $B$ (identical of mass $m = 2$ kg) rest on a rough surface, with coefficient of restitution $\mu = 0.40$, inclined at an angle of $\phi = 20^\circ$. Block $A$ has an initial speed of $v_{A,0} = 4$ m/s up the surface, while block $B$ is initially at rest, a distance $d = 0.50$ m up the surface. The coefficient of restitution between blocks $A$ and $B$ is $e = 0.75$. Find the distance up the surface traveled by block $B$ before it comes to rest (hint: use work-energy to describe the motion of the blocks and conservation of momentum to describe the collision).

Problem 29:
In the system shown to the right, the plunger (mass $m_1 = 2$ kg) is pulled back to compress the spring (stiffness $k$) by a distance $\delta = 5$ cm and released from rest. As the plunger returns to its unstretched position, it impacts a ball of mass $m_2 = 6$ kg with a coefficient of restitution $e = 2/3$. Finally, the plane of motion is inclined at an angle $\phi = 30^\circ$ and is assumed to be frictionless.

a) If the ball travels a distance $\ell = 1$ m, find the stiffness of the spring $k$;

b) Using the stiffness that you determined above, find the required initial compression of the spring so that the ball travels a distance $d = 0.5$ m up the plane.

Solution:
The response of this system can be broken down into three phases. The first describes the return of the plunger from its initial compressed position at time $t = t_0$, back to the point an instant before it contacts the ball at time $t = t_1^-$. The second phase describes the instantaneous collision between the plunger and ball, and spans from $t = t_1^- \rightarrow t = t_1^+$. In the final phase of the motion the ball moves up the inclined plane, reaching its maximum travel at time $t = t_2$. Over the first and third phase the system is conservative and we apply conservation of energy to relate the response at the beginning and end of each phase. The middle phase involves a collision, during which we apply conservation of momentum and the given coefficient of restitution.

During the initial phase of the motion the total energy of the system is conserved, so that

$$T_0 + V_0 = T_1^- + V_1^-, \quad \text{with:}$$

$$T_0 = 0, \quad T_1^- = \frac{1}{2}m_1 v_{1,1}^2, \quad V_0 = \frac{1}{2}k \delta^2 - m_1 g \delta \sin \phi, \quad V_1^- = 0.$$

In this, $v_{1,1}^-$ represents the velocity of the plunger just before impact. Solving for this velocity in terms of $\delta$ we obtain:

$$v_{1,1}^- = \sqrt{\delta \left( \frac{k}{m_1} \delta - 2g \sin \phi \right)}.$$
During the second phase of motion the linear momentum of the system is conserved, so that:

\[ m_1 v_{1,-} + m_2 v_{2,-} = m_1 v_{1,+} + m_2 v_{2,+}, \]

where \( v_{2,-} \) represents the velocity of the ball just before impact, equal to zero, while \( v_{1,+} \) and \( v_{2,+} \) described the velocities of the plunger and ball respectively just after the impact. This, together with the coefficient of restitution, defined as:

\[ e = \frac{v_{2,+} - v_{1,+}}{v_{1,-} - v_{2,-}}, \]

can be used to solve for the post-collision velocity of the ball, yielding:

\[ v_{2,+} = \frac{m_1}{m_1 + m_2}(1 + e)v_{1,-}, \]

where \( v_{1,-} \) is known from the previous phase of the motion.

In the final phase the total energy of the ball is again conserved, so that \( T_{1,+} + V_{1,+} = T_{2,+} + V_{2} \). Note that in this phase we only consider the motion of the ball. Although the total energy of the plunger is also constant, the motion of the plunger no longer affects the dynamics of the ball. At the final state \( (t = t_2) \), the velocity is constant and the ball moves up the plane a distance \( z \), so that:

\[ T_{1,+} = \frac{1}{2}m_2 v_{2,+}^2, \quad T_2 = 0, \]
\[ V_{1,+} = 0, \quad V_2 = m_2 g z \sin \phi. \]

Therefore, solving for \( z \) in terms of the post-collision velocity we find that:

\[ z = \frac{v_{2,+}^2}{2g \sin \phi}. \]

Finally, we can relate the initial compression of the spring to the distance the ball travels, which yields:

\[ z = (1 + e)^2 \frac{m_2^2 \delta}{(m_1 + m_2)^2} \left( \frac{k \delta}{2m_1 g \sin \phi} - 1 \right). \]  

(1)

a) Using Eq. (1), with \( z = \ell \), we can solve for \( k \) as:

\[ k = \frac{2m_1 g \sin \phi}{\delta} \left( \frac{(m_1 + m_2)^2}{\ell} \frac{\ell}{m_1^2 \delta} + 1 \right). \]

For the numerical values given in the problem statement, we find that \( k = 4.56 \times 10^4 \) N/m

b) If the ball moves up the plane a distance \( d \), then the initial compression \( x_c \) can be determined from Eq. (1). Substituting in the value of \( k \) determined above, this reduces to:

\[ d = \ell x_c^2 - \frac{d \delta^2}{2} + \frac{(1 + e)^2 m_2^2 \delta}{(m_1 + m_2)^2} x_c (x_c - \delta), \]

which can be written as a quadratic equation in \( x_c \):

\[ 0 = [(m_1 + m_2)^2 + (1 + e)^2 m_1^2 \delta] x_c^2 - [(1 + e)^2 m_1^2 \delta^2] x_c - (m_1 + m_2)^2 \delta^2. \]
With the parameter values given, this can be solved for $x_c$ as:

$$x_c = 0.035 \text{ m}.$$  

---

**Problem 30:**

A block of mass $m$ slides down a rough inclined plane, with inclination $\phi$ and unknown coefficient of friction $\mu$. The block is released with initial speed $v_0$ down the plane and after it has traveled a distance $d$ the speed is measured to be $v_1$.

1. Find the coefficient of friction $\mu$.
2. How far does the block slide before it comes to rest?

---

**Solution:**

We can use the work-energy formulation to relate the motion of the block between times $t = t_0$ and $t = t_1$.

For this system a free-body diagram is shown to the right. In addition, relating $\hat{\mathbf{j}}$ to the directions normal and tangential to the surface:

$$\hat{\mathbf{j}} = -\sin \phi \hat{\mathbf{e}}_1 + \cos \phi \hat{\mathbf{e}}_2.$$

Therefore, linear momentum balance provides:

$$\sum \mathbf{F} = m \mathbf{a}_{c_1}$$

$$\left( f_r + mg \sin \phi \right) \hat{\mathbf{e}}_1 + \left( N - mg \cos \phi \right) \hat{\mathbf{e}}_2 = \dot{v} \hat{\mathbf{e}}_1.$$

Because there is no acceleration in the $\hat{\mathbf{e}}_2$ direction, the forces must balance in this direction, so that:

$$N = mg \cos \phi.$$

Therefore, assuming the block slides down the surface (as given in the problem statement), the friction force is:

$$f_r \hat{\mathbf{e}}_1 = -\mu mg \cos \phi \hat{\mathbf{e}}_1.$$

Applying the equations of work-energy to this system as it slides a distance $x$ (obtaining a velocity $v$):

$$\int_0^x \left\{ \left( f_r + mg \sin \phi \right) \hat{\mathbf{e}}_1 + \left( N - mg \cos \phi \right) \hat{\mathbf{e}}_2 \right\} \cdot (ds \hat{\mathbf{e}}_1) = \frac{m}{2} \left( v^2 - v_0^2 \right),$$

$$mg \left( -\mu \cos \phi + \sin \phi \right) x = \frac{m}{2} \left( v^2 - v_0^2 \right).$$
a) If, after sliding a distance $d$ the speed of the block is $v_1$, we can solve the above equation for $\mu$ as:

$$\mu = \frac{v_0^2 - v_1^2 + 2gd \sin \phi}{2gd \cos \phi}.$$ 

b) The total distance the block slides, say $x = x_f$ is reached as the final velocity vanishes, so that $v = 0$. Again the above equation can be solved to give:

$$x_f = \frac{v_0^2}{2g \left( \mu \cos \phi - \sin \phi \right)}.$$ 

Using the value of $\mu$ found in the part a, this can be written as:

$$x_f = \frac{d}{1 - \left( \frac{v_1}{v_0} \right)^2}.$$ 

Problem 31:
As illustrated in the figure, the projectile, subject to only the influence of gravity, is launched with initial speed $v_0$ at an angle $\theta_0$. For what range of distances $\ell$ will the projectile clear the barrier of height $d$?

\[ \begin{align*} \text{Solution:} \\
\text{We identify the origin as } O \text{ and the projectile as } P, \text{ with the position of the projectile measured as:} \\
r_{PO} &= x(t) \hat{i} + z(t) \hat{j}, \\
\text{and the equations of motion for this system reduce to:} \\
\ddot{x} &= 0, \hspace{1cm} \ddot{z} = -g. \\
\text{Upon integrating with respect to time, these reduce to:} \\
\dot{x}(t) &= 0, \hspace{1cm} \dot{z}(t) = -g, \\
\dot{x}(t) &= \dot{x}(0), \hspace{1cm} \dot{z}(t) = -g t + \dot{z}(0), \\
x(t) &= \dot{x}(0) t + x(0), \hspace{1cm} z(t) = -\frac{g t^2}{2} + \dot{z}(0) t + z(0). \\
\text{Using the initial conditions:} \\
x(0) &= 0, \hspace{1cm} z(0) = 0, \hspace{1cm} \dot{x}(0) = v_0 \cos \theta, \hspace{1cm} \dot{z}(0) = v_0 \sin \theta, \\
\text{The position of the projectile reduces to:} \\
r_{PO} &= ((v_0 \cos \theta) t) \hat{i} + \left( -\frac{g \ell^2}{2} + (v_0 \sin \theta) t \right) \hat{j}. \\
\end{align*} \]
From this, we may solve for the time \( t^\star \) where \( z(t^\star) = d \) as:

\[
t^\star = \frac{(v_0 \sin \theta) \pm \sqrt{(v_0 \sin \theta)^2 - 2 \ g \ d}}{g}.
\]

Therefore there exist two times for which the projectile is at height \( d \), at which time the horizontal displacement is found to be:

\[
x(t^\star) = \frac{v_0 \cos \theta}{g} \left( (v_0 \sin \theta) \pm \sqrt{(v_0 \sin \theta)^2 - 2 \ g \ d} \right).
\]

So, for:

\[
\frac{v_0 \cos \theta}{g} \left( (v_0 \sin \theta) - \sqrt{(v_0 \sin \theta)^2 - 2 \ g \ d} \right) \leq \ell
\]

\[
\leq \frac{v_0 \cos \theta}{g} \left( (v_0 \sin \theta) + \sqrt{(v_0 \sin \theta)^2 - 2 \ g \ d} \right);
\]

the projectile clears the barrier.

**Problem 32:**

A girl throws a ball with an initial speed \( v_0 \) at an angle of \( \theta = 36.9^\circ = \sin^{-1} \frac{3}{5} \) from a height of \( d = 1.5 \) m. Find the initial speed so that the ball travels \( \ell = 10 \) m before striking the ground.

**Solution:**

we choose to describe the position of the ball using the coordinates \((x, z)\) which are measure the displacement of the ball in the \( \hat{i} \) and \( \hat{k} \) directions respectively. The initial conditions on the motion of the ball can be written as

\[
r_{BO}(0) = x(0) \hat{i} + z(0) \hat{k} = 0 \hat{i} + d \hat{k}, \quad x v_B = \dot{x}(0) \hat{i} + \ddot{z}(0) \hat{k} = v_0(\cos \theta \hat{i} + \sin \theta \hat{k}),
\]

while the final conditions, which occur when the ball strikes the ground at, say, some unknown time \( t_f \), are

\[
r_{BO}(t_f) = x(t_f) \hat{i} + z(t_f) \hat{k} = l \hat{i} + 0 \hat{k}.
\]

Although we may now use our expressions for the motion of particles in free-fall, it is more instructive to develop these equations from Newton’s laws. The only force acting on the particle is due to gravity, \(-mg\hat{k}\), and the acceleration is written as \( x a_B = \ddot{x} \hat{i} + \ddot{z} \hat{k} \). Thus we write the force-acceleration relationship as

\[
\sum F = m x a_B, \quad -mg \hat{k} = m(\ddot{x} \hat{i} + \ddot{z} \hat{k}),
\]

or, taking components in the \( \hat{i} \) and \( \hat{k} \) directions

\[
\begin{align*}
\ddot{x}(t) &= 0, & \dddot{x}(t) &= -g, \\
\ddot{z}(t) &= \dot{x}(0), & \dddot{z}(t) &= -gt + \dot{z}(0), \\
x(t) &= \dot{x}(0)t + x(0), & z(t) &= -\frac{1}{2} \dddot{z}(t)^2 + \dot{z}(0)t + z(0).
\end{align*}
\]
Finally, substituting in our initial and final conditions

\[
\begin{align*}
\dot{x}(t) &= v_0 \cos \theta, & \dot{z}(t) &= -gt + v_0 \sin \theta, \\
x(t) &= (v_0 \cos \theta)t, & z(t) &= -\frac{1}{2}t^2 + (v_0 \sin \theta)t + d, \\
x(t_f) &= (v_0 \cos \theta)t_f = l, & z(t_f) &= -\frac{1}{2}t_f^2 + (v_0 \sin \theta)t_f + d = 0.
\end{align*}
\]

From the two equations governing the position at time \(t_f\), we find that

\[
t_f = \frac{l}{v_0 \cos \theta},
\]

or substituting into the equation for \(z(t_f)\)

\[
0 = -\frac{g}{2} \left( \frac{l}{v_0 \cos \theta} \right)^2 + (v_0 \sin \theta) \frac{l}{v_0 \cos \theta} + d,
\]

which can be solved for \(v_0\) to yield

\[
v_0 = \sqrt{\frac{gl^2}{2 \cos \theta (l \sin \theta + d \cos \theta)}} = 9.32 \text{ m/s}
\]

---

**Problem 33:**

A crate of mass \(m\) is being carried on the back of the flat-bed truck. The coefficient of friction between the crate and the bed is \(\mu\) while the gravitational acceleration is \(g\). In addition, the truck is traveling on a hill inclined at an angle of \(\phi\).

- a) Find the maximum deceleration so that the crate does not hit the cab of the truck;
- b) What is the maximum angle \(\phi = \phi_{\text{max}}\) of the hill so that the crate does not slide forward when the truck is traveling at constant velocity?

**Solution:**

a) Let us assume a basis with \(\mathbf{i}\) directed along the path of the truck, and \(\mathbf{k}\) normal to the surface, so that \(x\) measures the position of the crate with respect to the origin. In addition, we let \(y\) represent the position of the truck with respect to the origin. Note that \(\mathbf{i}\) is at an inclination \(\phi\) with respect to “horizontal”. Clearly, the acceleration of the crate is simply \(\mathbf{a}_c = \ddot{x} \mathbf{i}\). The acceleration of the truck, which is assumed known, can be expressed as \(\mathbf{a}_T = \ddot{y} \mathbf{i} = a \mathbf{i}\).

Three forces act upon the crate: the gravitational force, the normal force, and the frictional force. As a side note, these three types of forces also act on the truck. Although when considering the truck (which we are not in this problem, only the crate) you might be tempted to include the force that is stopping the truck (from the brakes). However, in reality this is precisely the role of the frictional force. After all, “brakes stop your wheels, but your wheels stop your car.”

Application of Newton’s second law to the crate yields:

\[
\sum \mathbf{F} = m \ddot{x} \mathbf{a}_c,
\]

\[
(mg \sin \phi - f) \ddot{x} + (N - mg \cos \phi) \ddot{k} = m \ddot{x} \mathbf{i}.
\]
We immediately see that:

\[ N = mg \cos \phi, \]
\[ f = m(g \sin \phi - \ddot{x}) \]

b) The crate will not hit the cab of the truck if (and only if) it remains stationary with respect to the truck, so that \( x(t) = y(t) \) and therefore \( \ddot{x} = \ddot{y} = a \). Therefore, the frictional force is sufficiently large enough to overcome the decceleration and the component of gravity along the bed. However, \( |f| \leq \mu N \). Therefore:

\[ f = m(g \sin \phi - \ddot{x}) \leq \mu N, \]
\[ m(g \sin \phi - a) \leq \mu mg \cos \phi, \]
\[ mg(\sin \phi - \mu \cos \phi) \leq a. \]

Thus the decceleration (which is \(-a\)) must be less than \( mg(\mu \cos \phi - \sin \phi) \).

c) Again, we require that the relative velocity between the crate and the truck be zero, so that, in this situation \( \ddot{x} = 0 \). As above:

\[ f = m(g \sin \phi - \ddot{x}) \leq \mu N, \]
\[ mg \sin \phi \leq \mu mg \cos \phi, \]
\[ \tan \phi \leq \mu, \]

or finally \( \phi \leq \arctan \mu \).

---

**Problem 34:**

A mass \( m \), under the influence of gravity, is swinging around a vertical axis at the end of a massless string of length \( l \). The angular speed of the mass is given as \( \omega = \text{constant} \). If the mass is supported by a frictionless platform that is located at a height \( h \) below the support for the string:

a) find the tension in the string in terms of \( \omega \), \( m \), and \( l \);

b) what is the maximum value of \( \omega \) (for fixed \( m \) and \( l \)) so that the mass remains in contact with the table?

Hint: use \( \hat{k} \) as well as the \( \hat{e}_r \) and \( \hat{e}_\theta \) directions.

**Solution:**

We choose cylindrical coordinates \((\hat{e}_r, \hat{e}_\theta, \hat{k})\), where position of the mass with respect to the center of the platform, say \( C \), is \( \mathbf{r}_{PC} = r \hat{e}_r \). \( \hat{k} \) is directed upwards, so that, if \( O \) is the point of support for the string

\[ \mathbf{r}_{PO} = \sqrt{l^2 - h^2} \hat{e}_r - h \hat{k}. \]

As a result, with constant angular speed \( \omega \), the acceleration of the mass is

\[ \mathbf{a}_r = \left(-\omega^2 \sqrt{l^2 - h^2}\right) \hat{e}_r. \]
The total force acting on the mass is composed of the normal force, the force due to gravity, and the tension in the string

\[ \sum \mathbf{F} = (-T \cos \theta) \hat{\mathbf{e}}_r + (N + T \sin \theta - mg) \hat{\mathbf{k}}, \]

where, assuming that the mass is on the table, \( \sin \theta = \frac{h}{l} \). So, Newton’s second law can be written as

\[ \sum \mathbf{F} = m \mathbf{a}_r, \quad \left(-T \frac{\sqrt{l^2 - h^2}}{l}\right) \hat{\mathbf{e}}_r + \left(N + T \frac{h}{l} - mg\right) \hat{\mathbf{k}} = m \left(-\omega^2 \sqrt{l^2 - h^2}\right) \hat{\mathbf{e}}_r, \]

or, taking components in the \( \hat{\mathbf{e}}_r \) and \( \hat{\mathbf{k}} \) directions

\[ T \frac{\sqrt{l^2 - h^2}}{l} = m\omega^2 \sqrt{l^2 - h^2}, \quad N + T \frac{h}{l} - mg = 0. \]

a) The tension in the string is found by solving the above equations for \( T \), to yield

\[ T = ml\omega^2. \]

b) Finally, the normal force required to keep the ball on the table is

\[ N = mg - T \frac{h}{l} = m \left(g - h\omega^2\right), \]

So, if \( N < 0 \), this implies that the mass will lift off the table and swing freely. The critical angular speed is obtained by setting \( N = 0 \), to yield

\[ \omega_{cr} = \sqrt{\frac{g}{h}} \]

Interestingly, the tension is independent of \( h \) and the critical speed is independent of \( l \).

---

**Problem 35:**

A cannonball is fired with an initial speed of 50 m/s. Find the maximum distance the ball can travel if its height never exceeds 25 m.

---

**Problem 36:**

The upper block (with mass \( m \)) sits on a rough surface with coefficient of friction \( \mu \). If the block suspended by the cable is assumed to have mass \( \alpha m \), with \( \alpha > 1 \), find:

a) the acceleration of the upper block;

b) the minimum value of \( \alpha \), defined as \( \alpha_0 \), so that the suspended block falls;

c) what happens if \( \alpha < \alpha_0 \)?
Problem 37:
In the pulley system shown, the mass of each block is identical, say \( m \). If a constant force \( F \) pushes up on block \( B \), find the speed of each block after \( B \) has moved down a distance \( l \) (assume that \( F < mg \) so that motion occurs in the downward direction) if the system starts from rest.

Solution:
a) Let \( x \) measure the position of \( A \) (positive to the left) and \( y \) denote the position of \( B \) (positive down). Because we assume that all forces are conservative (i.e. no friction), the total energy of the system is conserved throughout the motion, so that \( T_1 + V_1 = T_2 + V_2 \). In addition, we assume that the potential energy at state 1 is zero—the potential energy datum is located at the initial state. Consequently:

\[
\begin{align*}
T_1 &= 0, \\
V_1 &= 0, \\
T_2 &= \frac{m}{2} (\dot{x}^2 + \dot{y}^2), \\
V_2 &= (F - mg)y,
\end{align*}
\]

Finally, we see that the string enforces the following constraint:

\[
2x + y = \text{constant}, \\
2\dot{x} + \dot{y} = 0.
\]

So \( \Delta E = 0 \) provides:

\[
0 = \frac{m}{2} \left( \left( -\frac{\dot{y}}{2} \right)^2 + \dot{y}^2 \right) + (F - mg)y.
\]

So solving for \( \dot{x} \) and \( \dot{y} \), after block \( B \) has fallen a distance \( l \), the speed of each block is:

\[
\begin{align*}
\dot{x} &= -\sqrt{\frac{2}{5} \left( g - \frac{F}{m} \right)} l, \\
\dot{y} &= \sqrt{\frac{8}{5} \left( g - \frac{F}{m} \right)} l.
\end{align*}
\]

Problem 38:
Two blocks, each of mass \( m \), are connected by a spring (with spring constant \( k \)). If the blocks are placed on a smooth surface, the spring is initially stretched by a distance \( d \), and released from rest, find:

a) the speed of each block when the spring is unstretched;

b) the maximum compression in the spring.

Solution:
We combine both conservation of energy and conservation of momentum to approach this problem. Clearly, the system is conservative so that $\Delta E = 0$, and because no external forces act in the horizontal direction, we have $\Delta \mathbf{L} \cdot \mathbf{i} = 0$, where $\mathbf{i}$ is directed along the ground. Let $x_1$ and $x_2$ denote the position of masses 1 and 2, respectively, as measured from some origin fixed in the ground. Thus:

\[ T = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2), \]
\[ V = \frac{k}{2} ((x_2 - x_1) - l_0)^2, \]

where $l_0$ is the unstretched length of the spring, so that $(x_2 - x_1) - l_0$ represents the stretch in the spring. Considering the linear momentum of the system, which is conserved:

\[ \mathbf{L} = (m\dot{x}_1) \mathbf{i} + (m\dot{x}_2) \mathbf{i}, \]

so that, because the system starts from rest, $\mathbf{L} = \mathbf{0}$. As a result, $\dot{x}_1 = -\dot{x}_2$, and conservation of energy provides:

\[ \frac{k}{2} d^2 = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) + \frac{k}{2} ((x_2 - x_1) - l_0)^2, \]
\[ \frac{k}{2} d^2 = m\dot{x}_1^2 + \frac{k}{2} ((x_2 - x_1) - l_0)^2. \]

a) If the spring is uncompressed, $(x_2 - x_1) - l_0 = 0$, so that:

\[ \frac{k}{2} d^2 = m\dot{x}_1^2, \]

or:

\[ \dot{x}_1 = \sqrt{\frac{k}{2m} d}. \]

b) When the spring is at maximum compression, the relative velocity between the two masses is zero, so that:

\[ \frac{k}{2} d^2 = \frac{k}{2} ((x_2 - x_1) - l_0)^2, \]

or the maximum compression is simply $d$, identically equal to the initial stretch.

---

**Problem 39:**

A massless rod of length $l$ is connected to a particle with mass $m$, and is released from rest in a horizontal position. Find the velocity of the particle (with mass $2m$), which is initially at rest on the ground, after the collision. Assume that the coefficient of restitution is $\varepsilon = 0.8$ and gravity is the only external force.

**Solution:**

Consider two regimes of the motion: falling of the pendulum, in which the energy is conserved, and impact, which conserves linear momentum. Thus, in phase one, $\Delta E = 0$, so that $T_1 + V_1 =$
\( T_2 + V_2 \). Since the system is started from rest, the initial kinetic energy is zero, and, in addition, we define the potential energy at the surface to vanish. Consequently, we find:

\[
\begin{align*}
T_1 &= 0, \\
V_1 &= mgl, \\
T_2 &= \frac{m}{2}v_{1,2}^2, \\
V_2 &= 0,
\end{align*}
\]

where \( v_{1,2} \) is the speed of mass 1 at state 2, just before impact. Therefore, we find:

\[ v_{1,2} = \sqrt{2gl}. \]

During the impact, the linear momentum is conserved, i.e. \( \Delta L = 0 \). If we define \( v_{i,j} \) to be speed of mass \( i \) at state \( j \), this conservation law can be expressed as:

\[
(m)v_{1,2} + (2m)v_{2,2} = (m)v_{1,3} + (2m)v_{2,3},
\]

where state 3 is just after the impact. Because mass 2 is initially at rest, \( v_{2,2} = 0 \), which results in:

\[ v_{1,2} = v_{1,3} + 2v_{2,3}. \]

Now utilizing the coefficient of restitution:

\[ \varepsilon = \frac{v_{2,3} - v_{1,3}}{v_{1,2} - v_{2,2}}, \]

and solving for \( v_{2,3} \), we obtain:

\[
\begin{align*}
v_{2,3} &= \frac{1 + \varepsilon}{3}v_{1,2}, \\
&= \frac{1 + \varepsilon}{3}\sqrt{2gl},
\end{align*}
\]

so that the velocity of mass 2 after the collision is:

\[ \varepsilon v_{2,3} = \left( -\frac{1 + \varepsilon}{3}\sqrt{2gl} \right) \mathbf{i}. \]

---

**Problem 40:**

Two blocks, one of mass \( m_1 \), the second of mass \( m_2 \), are connected by a spring (with spring constant \( k \)) and resting on a smooth surface. If the spring is initially stretched by a distance \( d \), and the blocks released from rest, find:

a) the speed of each block when the spring is unstretched;

b) the maximum compression in the spring.

**Solution:**

39
We note that in this system both energy and linear momentum are conserved. The first conservation law results from the conservative nature of the spring force and lack of friction, while the latter stems from the absence of external forces in the \( \hat{i} \) direction (we neglect the dynamical equations in the \( \hat{j} \) direction). These laws can be written as:

\[
\Delta E = 0 \rightarrow T_1 + V_1 = T_2 + V_2, \\
\Delta L = 0 \rightarrow 0 = m_1 v_{1,2} + m_2 v_{2,2},
\]

where \( v_{i,2} \) represents the speed of mass \( i \) at state 2, and \( s \) is the stretch in the spring at this final state. We again point out that initially the system is at rest, so that \( v_{i,1} = 0 \).

a) If the spring is unstretched, \( s = 0 \), and our conservation laws reduce to:

\[
\frac{k}{2} d^2 = \frac{1}{2} \left( m_1 v_{1,2}^2 + m_2 v_{2,2}^2 \right), \\
0 = m_1 v_{1,2} + m_2 v_{2,2},
\]

or solving for \( v_{1,2} \) and \( v_{2,2} \), we obtain:

\[
v_{1,2} = \sqrt{\frac{m_2}{m_1} \left( \frac{k d^2}{m_1 + m_2} \right)}, \quad v_{2,2} = \sqrt{\frac{m_1}{m_2} \left( \frac{k d^2}{m_1 + m_2} \right)}.
\]

b) If the spring is maximally compressed at say \( s = \delta_{\text{max}} \), then the relative velocity between the masses is zero, and therefore from conservation of linear momentum, both velocities are zero. Consequently, conservation of energy implies:

\[
\frac{k}{2} \delta_{\text{max}}^2 = \frac{k}{2} \delta_{\text{max}}^2, \\
\frac{d}{\delta_{\text{max}}} = \delta_{\text{max}}.
\]

that is, the maximum compression is equal to the initial extension.

---

**Problem 41:**
A particle with mass \( m \) is initially used to compress a spring a distance \( d \). The mass is released from rest and, at some later time, impacts a second ball (with mass \( 2m \)), which is initially at rest on the ground. Find the velocity of both balls after the collision if the coefficient of restitution is \( \varepsilon = 0.8 \). Assume that the surface is frictionless.
Problem 42:
A block of mass $m$ rests on the surface of a cart, having mass $2m$. If the spring (with spring constant $k$), which is connected to both the block and the cart is initially compressed a distance $d$, and the system is released from rest, find:

a) the speed of the block with respect to the cart when the spring is unstretched;

b) the maximum compression in the spring.

Problem 43:
The upper block (with mass $m$) sits on a rough surface with coefficient of friction $\mu$. If the block suspended by the cable is assumed to have mass $\alpha m$, with $\alpha > 1$, find:

a) the acceleration of the upper block;

b) the minimum value of $\alpha$, defined as $\alpha_0$, so that the suspended block falls;

c) what happens if $\alpha < \alpha_0$?

Problem 44:
A ball of mass $m$ with an initial speed $v_0$ slides down a frictionless surface from a height $h$, and strikes a massless spring. Find:

a) the maximum compression of the spring;

b) the height at which the ball reaches after contact is lost with the spring;

c) if surface friction is included, will the maximum compression of the spring be greater than or less than the frictionless case? Explain, although no explicit calculations are necessary.
**Problem 45:**
In the system shown to the right, the blocks have mass $m_A = 10 \text{ kg}$ and $m_B = 2 \text{ kg}$. If all of the surfaces are smooth determine the acceleration of block $A$ and the tension in the cord.

![Diagram of blocks](image)

**Problem 46:**
In the system shown to the right, the system is released from rest at the unstretched position of the spring.

a) What is the maximum stretch in the spring during the motion;

b) What is the maximum velocity obtained by the masses?

**Problem 47:**
The two spheres, each with mass $m = 1 \text{ kg}$ collide with the initial velocities as shown in the figure. If the contact normal is inclined at $45^\circ$ and the coefficient of restitution is $e = 0.75$, find the final velocity of each ball.

![Diagram of spheres](image)

$\vec{v}_1 = (-0.50 \text{ m/s}) \hat{i}$

$\vec{v}_2 = (0.50 \text{ m/s}) \hat{j}$

3 Planar Dynamics
Problem 48: (20 pts.)
A sled of mass $m_1$ is pulled to the right by a vehicle of mass $m_2 = 1000$ kg along a rough surface with coefficient of friction $\mu = 0.25$. In terms of the mass of the sled $m_1$, find the maximum driving force that can be applied from the rear wheels so that contact remains between the front wheels of the vehicle and the surface, i.e., the vehicle does not tip over. Assume that the width of the sled is such that it never tips. Note that all distances are given in terms of $h = 1.50$ m.

Solution:
We define the displacement of the system as $x$ as described in the figure. Therefore the acceleration of both the vehicle and the sled can be written as $\dot{x}^a_G = \dot{x}^a_P = \ddot{x}^i$.

Then, free body diagrams for both the sled and the vehicle can be constructed as shown to the right. The tension in the cable connecting the sled and the vehicle has (unknown) magnitude $T$ while the driving force applied at the rear wheels is $f_d^i$. Notice that the normal force acting on both wheels has been included. However, when the vehicle is just about to tip the normal force at the front wheels vanishes, so that $N_3 = 0$. This is the condition that determines the maximum driving force $f_{d,\text{max}}$.

The equations of motion will be developed by applying linear momentum balance separately on both the sled and the vehicle, and angular momentum balance about $G$ on the vehicle. Because the sled is assumed to never tip, we can safely neglect angular momentum balance on this component.

Linear momentum balance on the sled provides
\[
\sum \mathbf{F} = (T - \mu N_1) \dot{x}^i + (N_1 - m_1 g) \ddot{x}^j = m_1 \ddot{x}^i = m_1 \dot{x} \mathbf{a}_P,
\]
while linear and angular momentum balance on the vehicle yield
\[
\sum \mathbf{F} = (-T + f_d) \dot{x}^i + (N_2 + N_3 - m_2 g) \ddot{x}^j = m_2 \ddot{x}^i = m_2 \dot{x} \mathbf{a}_G,
\]
\[
\sum \mathbf{M}_G = \left( T h + f_d \frac{h}{2} + N_3 (2 h) - N_2 h \right) \dot{k} = 0.
\]

From the linear momentum balance equations in the $\dot{j}$ direction we find that $N_1 = m_1 g$ and $N_2 = m_2 g - N_3$, so that when the vehicle is about to tip $N_3 = 0$, and $N_2 = m_2 g$. Then, the
remaining scalar equations are

\[ T - \mu m_1 g = m_1 \ddot{x}, \]
\[ -T + f_{d,\text{max}} = m_2 \ddot{x}, \]
\[ \frac{h}{2} (2T + f_{d,\text{max}} - 2m_2 g) = 0. \]

Finally, solving these for \( f_{d,\text{max}} \) we find that

\[ f_{d,\text{max}} = \frac{2m_2 g \left( (1 - \mu) m_1 + m_2 \right)}{3m_1 + m_2} = \left[ \frac{0.75m_1 + 1000 \text{ kg}}{3m_1 + 1000 \text{ kg}} \right] 19620 \text{ N}. \]

Although this was not part of the problem statement, from the above equations of motion the tension in the cable and the acceleration of the system when the vehicle is on the verge of tipping can be solved as

\[ T = \frac{(2 + \mu) m_1 m_2 g}{3m_1 + m_2}, \quad \ddot{x} = \frac{(2m_2 - 3\mu m_1) g}{3m_1 + m_2}. \]

Note that the acceleration of the block is positive and the block continues to slide to the right provided \( m_1 < \frac{(2m_2)}{(3\mu)}. \)

**Problem 49: (20 pts.)**

The two masses are supported by cables of equal length \( \ell = 40 \text{ cm} \). The left mass is released from rest in the horizontal position, falls down and impacts the second mass, with \( m_2 = 2 \text{ kg} \), which is initially stationary in its vertical equilibrium position. Find the coefficient of restitution \( e \) between the two masses and the mass of the left mass \( m_1 \) so that after the impact the left mass remains stationary and the right mass swings through a maximum angle of \( \phi = 60^\circ \).

Note: at \( t = 0 \)
\[ \phi_1(0) = -\pi/2 \]
\[ \phi_2(0) = 0 \]
and at \( t = t_3 \)
\[ \phi_1(t_3) = 0 \]

**Solution:**

The response of this system can be described in three phases, in which energy is conserved as the left pendulum falls \( (t_0 < t < t_1) \), conservation of momentum during the impact \( (t_1 = t_{\text{impact}} < t < t_3) \), and conservation of energy again as the right pendulum swings up \( (t_2 < t < t_3) \).

With the coordinates \( \phi_1 \) and \( \phi_2 \) defined as shown in the figure, the speed of each mass can be expressed as

\[ v_1 = \ell \dot{\phi}_1, \quad v_2 = \ell \dot{\phi}_2. \]

Finally, the speed of mass \( i \) at time \( t_j \) is denoted as \( v_{i,j} \equiv v_i(t_j) \), so that

\[ v_{1,0} = v_{2,0} = 0, \quad v_{2,1} = 0, \quad v_{1,2} = 0, \quad v_{1,3} = v_{2,3} = 0. \]
In general, the kinetic and potential energy of this system can be defined as
\[ T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2, \quad V = m_1 g \ell \left(1 - \cos \phi_1\right) + m_2 g \ell \left(1 - \cos \phi_2\right), \]
where the potential energy is assumed to vanish at the lower position of the pendula. Therefore, the energy \( E = T + V \) at each time can be expressed as
\[
\begin{align*}
E_0 &= T_0 + V_0 = m_1 g \ell, \\
E_1 &= T_1 + V_1 = \frac{1}{2} m_1 v_{1,1}^2, \\
E_2 &= T_2 + V_2 = \frac{1}{2} m_2 v_{2,2}^2, \\
E_3 &= T_3 + V_3 = m_2 g \ell \left(1 - \cos 60^\circ\right) = m_2 g \frac{\ell}{2}.
\end{align*}
\]
Therefore across the initial and final stages of the motion, during which energy is conserved
\[
\begin{align*}
v_{1,2} &= \sqrt{2 g \ell}, \\
v_{2,2} &= \sqrt{g \ell}.
\end{align*}
\]
During the impact the linear momentum is conserved, so that
\[
m_1 v_{1,1} + m_2 v_{2,1} = m_1 v_{1,2} + m_2 v_{2,2} \quad \rightarrow \quad m_1 v_{1,1} = m_2 v_{2,2}.
\]
In addition, the coefficient of restitution is defined as
\[
e = \frac{v_{2,2} - v_{1,2}}{v_{1,1} - v_{2,1}} \quad \rightarrow \quad e = \frac{v_{2,2}}{v_{1,1}}
\]
Substituting in for the values of \( v_{2,2} \) and \( v_{1,1} \) found earlier
\[
e = \frac{\sqrt{g \ell}}{\sqrt{2 g \ell}} = \frac{1}{\sqrt{2}}, \quad m_1 = m_2 \frac{\sqrt{g \ell}}{\sqrt{2 g \ell}} = \frac{m_2}{\sqrt{2}} = \sqrt{2} \text{ kg}
\]

**Problem 50:**
In the system shown to the right, a cable is wrapped around the uniform disk (mass \( m = 2 \text{ kg} \), radius \( r = 0.10 \text{ m} \)) with one end attached to the ceiling and the other end supported by a spring with stiffness \( k = 4000 \text{ N/m} \). Assume the cable does not slip on the disk. A second mass \( (m = 4 \text{ kg}) \) is also attached to the disk as shown. If the center of the disk is pulled down a distance \( x(0) = d = 0.05 \text{ m} \) (as measured from the unstretched position of the spring) and released from rest:

a) find the velocity of the disk when it returns to the unstretched position of the spring;

b) find the maximum velocity of the center of the disk.
Problem 51:
A block of mass \( m = 4 \) kg slides on a smooth surface. A rigid bar with mass \( m = 1 \) kg and length \( \ell = 0.25 \) m is mounted on the block and is supported by a spring of stiffness \( k = 1000 \) N/m, which is unstretched in the vertical position. Find the steady-state angle \( \theta \) made by the bar with respect to its vertical position when the block is pushed by a constant force \( F i = 15 \) N \( i \). The spring is attached to a collar so that as the bar moves the spring always remains horizontal.

\[ g \]

Problem 52:
In the system shown to the right, a cable is wrapped around the disk (mass \( m \) radius \( r \)) with one end attached to the ceiling and the other end supported by a spring with stiffness \( k \). Assume the cable does not slip on the disk. If the center of the disk is pulled down a distance \( x(0) = d \) (as measured from the unstretched position of the spring) and released from rest. Find the maximum velocity of the center of the disk.

\[ g \]

Problem 53:
A uniform rod (length \( \ell = 25 \) cm, mass \( m = 1 \) kg) leans against a corner at angle \( \theta = 53.13^\circ \) and is released from rest. The vertical surface is smooth (\( \mu = 0 \)) but the horizontal surface is rough, with coefficient of friction \( \mu \).

a) Find the maximum coefficient of friction for which the rod remains stationary, leaning against the corner;

b) For \( \mu = 0.25 \), find the initial angular acceleration of the rod.
Problem 54:
In the system shown to the right, a cable is wrapped around the disk (mass \( m \), radius \( r \)) with one end attached to the ceiling and the other end supported by a spring with stiffness \( k \). Assume the cable does not slip on the disk. In addition, a force \( -F \hat{j} \) is applied to the center of the disk. If the center of the disk is pulled down a distance \( d \) (as measured from the unstretched position of the spring) and released from rest:

a) find the angular acceleration of the disk at this instant;

b) what is the tension in the cable on the left at this instant?

Solution:
We mark the points \( A \) and \( B \) as the contact points between the cable and the disk on the left and right respectively. In addition, we identify the three coordinates \( x, y, \) and \( \theta \) as shown in the figure. Note that \( x \) describes the displacement of the mass center \( G \), which is not equal to \( y \), the displacement at \( B \).

Because the cable does not slip on the disk, \( \mathbf{v}_A = 0 \). The velocities of \( G \), the mass center, and \( B \), the right contact point, may be found as:

\[
\mathbf{v}_G = \mathbf{v}_A + \omega_{D/\hat{F}} \times \mathbf{r}_{GA},
\]

\[
-\dot{x} \hat{j} = \dot{\theta} \hat{k} \times \mathbf{r}_{i},
\]

\[
= r \dot{\theta} \hat{j},
\]

\[
\mathbf{v}_B = \mathbf{v}_A + \omega_{D/\hat{F}} \times \mathbf{r}_{BA},
\]

\[
-\dot{y} \hat{j} = \dot{\theta} \hat{k} \times 2r \hat{i},
\]

\[
= 2r \dot{\theta} \hat{j}.
\]

Therefore, these three coordinates are related as:

\[
x = -r \theta, \quad y = -2r \theta.
\]

An appropriate free-body diagram for the disk is shown to the right. We define the tension in the left cable as \( T \hat{j} \), while the force on the right is equal to the spring force \( k y \hat{j} \). Notice that I have neglected gravity, but it could easily be incorporated into the external force \( -F \hat{j} \).

Applying angular and linear momentum balance to this system yields the equations:

\[
\sum \mathbf{F} = m \mathbf{\ddot{r}}_G: \quad T \hat{j} + k y \hat{j} - F \hat{j} = -m \dot{x} \hat{j},
\]

\[
\sum M_G = I_G \alpha_{D/\hat{F}}: \quad k y r \hat{k} - T r \hat{k} = I_G \ddot{\theta} \hat{k}.
\]

a) These equations, together with the coordinate relations defined above, can be used to solve for \( \dot{\theta} \), the angular acceleration, in terms of the displacement \( x \), as:

\[
\dot{\theta} = \frac{2}{3 m r} (4k x - F).
\]
Therefore, with an initial displacement \( x(0) = d \), the initial angular acceleration is:

\[
\alpha_{\theta/x} = \left( \frac{2}{3m} \right) (4k d - F) \hat{k}.
\]

b) Going back to the equations, we can also solve for the tension in the left cable as a function of the displacement of the center of the disk:

\[
T = \frac{2k x + F}{3},
\]

and the initial tension in the cable is:

\[
T \hat{j} = \frac{2k d + F}{3} \hat{j}.
\]

**Problem 55:**

A block \((m = 2 \text{ kg}, d = 0.20 \text{ cm})\) is subject to a force \( F \hat{e}_1 \) and lies on a rough inclined plane, with inclination \( \phi = 30^\circ \) and coefficient of friction \( \mu \).

a) What is the maximum coefficient of friction for which the block slides with \( F = 0 \)?

b) With \( \mu = 0.25 \), find the acceleration of the block as a function of \( F \).

c) What forces act at the contact points?

**Solution:**

We begin by defining the displacement of \( G \), the mass center of the block, as \( x \hat{e}_1 \), so that the acceleration of the block becomes \( x \hat{a} = \ddot{x} \hat{e}_1 \).

A free body diagram for this system is shown to the right. Notice that the contact force acting on the left of the block is distinct from that acting on the right. In addition, the linear and angular acceleration of the block are:

\[
\hat{x} \alpha_g = \ddot{x} \hat{e}_1, \quad \alpha_{\theta/x} = 0.
\]

For this system, in terms of the \((\hat{e}_1, \hat{e}_2)\) directions the gravitational force becomes:

\[
-mg \hat{j} = mg \sin \phi \hat{e}_1 - mg \cos \phi \hat{e}_2.
\]

Also, if the block is sliding, the \(i^{th}\) friction force \( f_i \) is related to the normal force \( N_i \hat{e}_2 \):

\[
f_i = -\mu N_i \text{sgn}(\dot{x}).
\]
Linear momentum balance on this system yields:
\[ \sum F = ma, \]
\[ (F + f_1 + f_2 + mg \sin \phi) \hat{e}_1 + (N_1 + N_1 - mg \cos \phi) \hat{e}_2 = m \ddot{x} \hat{e}_1. \]

Likewise, angular momentum balance provides:
\[ \sum M_G = I_g \ddot{\alpha}, \]
\[ \left( -(2d) N_1 + (2d) N_2 + d (f_1 + f_2) - dF \right) \hat{k} = 0. \]

Therefore, these provide three scalar equations:
\[ F + (f_1 + f_2) + mg \sin \phi = m \ddot{x}, \]
\[ (N_1 + N_2) = mg \cos \phi, \]
\[ d \left( 2(N_2 - N_1) + (f_1 + f_2) - F \right) = 0. \]

a) Assuming the block is sliding down the plane, then:
\[ f_1 + f_2 = -\mu(N_1 + N_2) = -\mu mg \cos \phi. \]

Notice that we have not assumed that \( N_1 = N_2 \). Rather, only that we know the total normal force acting on the block. With this, for \( F = 0 \) the acceleration of the block can be solved as:
\[ \ddot{x} = g \left( \sin \phi - \mu \cos \phi \right). \]

The block continues to slide down the plane provided \( \ddot{x} \geq 0 \), which implies that:
\[ \sin \phi - \mu \cos \phi \geq 0, \quad \text{or} \quad \mu \leq \tan \phi. \]

For \( \phi = 30^\circ \) we find that the maximum coefficient of friction is:
\[ \mu_{\text{max}} = \frac{1}{\sqrt{3}} = 0.577. \]

b) More generally, if the block slides down the plane its acceleration is:
\[ \ddot{x} = \frac{F}{m} + g \left( \sin \phi - \mu \cos \phi \right). \]

Therefore, with \( \mu = 0.25 \) and the given system parameters:
\[ \ddot{x} = \frac{F}{(2 \text{ kg})} + \frac{4 - \sqrt{3}}{8} g = \frac{F}{(2 \text{ kg})} + 2.78 \text{ m/s}^2. \]

c) Finally, making use of the equation from angular momentum balance, we may solve for the normal forces to be:
\[ N_1 = \frac{mg \cos \phi}{2} \left( 1 - \frac{\mu}{2} \right) - \frac{F}{4}, \]
\[ N_2 = \frac{mg \cos \phi}{2} \left( 1 + \frac{\mu}{2} \right) + \frac{F}{4}. \]
Notice that as either $\mu$ or $F$ increases, the normal force acting on the right ($N_2$) increases as well, which is offset by an equal decrease of $N_1$ so that the total normal component $N_1 + N_2$ remains constant. However, $N_1$ and $N_2$ are clearly not necessarily equal. In fact, for:

$$F > mg \cos \phi (2 - \mu),$$

the normal force $N_1$ is predicted to be negative.

\[\text{Problem 56:}\]
A slender bar of mass $m$ and length $l$, is pinned at an interior point, say $O$, which is a distance $\delta l$ from $\bar{G}$. For $t < 0$ the bar is held stationary and horizontal by a string. If, at $t = 0$ the string is cut, find

\[\text{a) the angular acceleration at } t = 0^+, \text{ i.e. just after the string is cut. Note that at this time the angular acceleration is nonzero, although the angular velocity is still zero;}\]

\[\text{b) the reaction force acting on the bar at } O, \text{ at } t = 0^+;\]

\[\text{c) the value of } \delta \text{ so that the reaction force at } t = 0^+ \text{ is minimal.}\]

\[\text{Solution:}\]
\[\text{a) At time } t = 0^+, \text{ the only forces acting on the bar are that of gravity and the reaction force at the pin. In addition, although the angular acceleration of the bar is nonzero, the angular velocity is indeed zero—it has not had sufficient time to increase. As always, there are a number of different approaches to solving this problem.}\]

Let $(\hat{i}, \hat{j}, \hat{k})$ be a coordinate system fixed in the ground in the usual fashion. Angular momentum balance about $O$, about which the bar rotates, yields:

$$\sum M_O = I_O \alpha_{B/F},$$

$$-mg \delta \hat{k} = \left(\frac{ml^2}{12} + m(\delta l)^2\right) \alpha \hat{k}.\]

So solving for $\alpha$, we find the angular acceleration to be:

$$\alpha_{B/F} = \frac{12g \delta}{l(1 + 12\delta^2)} \hat{k}.\]

\[\text{b) To find the total reaction force at } O, \text{ say } F_{\text{reaction}} = F_x \hat{i} + F_y \hat{j}, \text{ we apply } \sum F = m \vec{a}_G. \text{ However, because the angular velocity is zero, this reduces to:}\]

$$\sum F = m (\alpha_{B/F} \times r_{GO}),$$

$$F_x \hat{i} + (F_y - mg) \hat{j} = m \left(\alpha \hat{k} \times (-\delta \hat{i})\right),$$

$$F_x \hat{i} + (F_y - mg) \hat{j} = -m \delta l \alpha \hat{j},$$

or finally solving for $F_{\text{reaction}}$:

$$F_{\text{reaction}} = \frac{mg}{1 + 12\delta^2} \hat{j}.\]
c) The minimum reaction force occurs when $\delta$ is maximal, i.e. $\delta = \frac{1}{2}$.

Problem 57:
A slender bar of mass $m$ and length $l$, is pinned at one end, say $O$. For $t < 0$ the bar is held stationary by a string and inclined at an angle $\phi$ to horizontal. If, at $t = 0$ the string is cut, find:

a) the angular acceleration at $t = 0^+$, i.e. just after the string is cut. Note that at this time the angular acceleration is nonzero, although the angular velocity is still zero;

b) the reaction force acting on the bar at $O$, at $t = 0^+$;

Problem 58:
A rod of mass $m$ and length $l$ is released from rest without rotating. When it falls a distance $d$ the end $A$ strikes the hook $S$, which produces a permanent connection. Determine the angular velocity $\omega$ of the rod after it has rotated 90°. Treat the rod’s weight during the impact as a nonimpulsive force.