On the $SO(2)$ Symmetric Deformation of Rotating Rings with Shear Deformation

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ABSTRACT
We study the $SO(2)$ symmetric deformation of a circular ring, modeled using the beamshell theory of Libai and Simmonds. The equations of motion are based on general partial differential equations governing the elastodynamics of geometrically exact rings, which have been formulated by Dempsey [6]. Thus, the formulation is valid for arbitrary pressure forcing and large deformations, although the results are tempered by a linear constitutive relation and an assumption of plane strain.

With the assumption that the deformation retains $SO(2)$ symmetry, the partial differential equations are reduced to a set of coupled ordinary differential equations. Within this restricted space of solutions, we study the existence and stability of relative equilibria and discuss the effects of constant hydrostatic pressure on the dynamical response. Specifically, interaction between inertial effects arising from rotational motion and the combined elastic and external pressure forces can produce unexpected behavior, including the existence of a nontrivial state which retains the symmetry, yet physically implies that material planes do not lie in the radial direction. Such a state is shown to affect the large-amplitude response of the system in a singular limit of the governing equations.

1 INTRODUCTION
The circular ring provides a simple, yet realistic model for many engineering systems, such as tires, ring stiffeners, and long cylindrical elements. In addition, the circular ring serves as a simple model for analyzing the effects of symmetry in structural systems. Finally, the derivation and analysis of equations of motion for such systems, including strings, rods, and shells, is one of the most widely studied areas of mechanics (see [2] and the references therein).

The dynamical behavior of such elastic bodies has been used extensively to study modal interactions when the linearized natural frequencies are commensurate. One of the earliest studies of modal interactions was performed by Goodier and McIvor [8], who studied the coupled response between a breathing and flexural mode of cylindrical and spherical shells, although our formulation differs from that of Goodier and McIvor through the inclusion of rotational inertia. Later, studies by Nayfeh et al. [17, 18] used the method of multiple scales to fully account for the nonlinear interaction between the modes. Natsiavas [16] has also used these singular perturbation techniques to study a 1:1 internal resonance within the ring as the rotational speed varies.

Several authors have studied rotating rings subject to extensibility and shear deformation and have found that both effects are necessary for an accurate description of the motion [3, 15]. Also, Pelchen and Wauer [19] have studied the effects of internal and external pressure on the free vibration frequencies.

General equations governing the elastodynamics of geometrically exact rings, with the rational inclusion of rotational motion, have been formulated by Dempsey [6], based on the work of Libai and Simmonds [14]. The formulation is valid for arbitrary pressure forcing and large deformations. In addition, the ring material is not assumed to be circumferentially “inextensible,” although the results are tempered by linear constitutive relations and an assumption of plane strain.

Shell theory is used to collapse the two-dimensional continuum of the ring onto a planar curve, and the dynamic
equations of motion describe the evolution of this curve and the material planes initially normal to it. In the undeformed configuration this curve—the shell—is coincident with the dynamic axis, which is the center-of-mass axis of the undeformed ring. The center-of-stiffness axis of the undeformed ring is the neutral axis identified in classical curved beam theory. The two axes are inherently eccentric from one another; whereas the neutral axis lies fractionally inside the midsurface axis, the dynamic axis lies fractionally outside the midsurface axis.

With the constitutive assumptions made in this paper, and in [6], the deformed shell is the deformed image of the dynamic axis, and the deformed center-of-stiffness axis is the deformed image of the neutral axis. As the system evolves, the shell and the deformed neutral axis remain eccentric. The coupled nature of ring dynamics is brought about by this eccentricity. Mathematically this is reflected in the governing partial differential equations of motion by the coupling between the evolution of the shell and the rotational motion of material planes. Regardless of the choice of kinematic coordinates, the equations of motion are either dynamically or statically coupled, or both, but cannot be completely uncoupled.

Part of our motivation for this work comes from the work of Dempsey and Gladwell [7], in which the breathing-mode dynamics of the ring exhibits a numerical instability. However, this instability only occurs with the inclusion of shear deformation [13]. If the deformation of the material normals is suppressed, the resulting numerical accuracy is that of the underlying spatial differencing scheme. Thus we hope to further illustrate effects of shear deformation on the resulting dynamic behavior of the shell.

If the assumption is made that the ring deforms but retains $O(2)$ symmetry, which physically requires that all normals deform identically, then the spatial derivatives vanish and the governing equations can be written as second-order ordinary differential equations in time. However, we find that a small parameter $\varepsilon$ multiplies a second-order derivative. Physically, $\varepsilon$ represents the nondimensional radius of gyration of material planes about the dynamic axis. Therefore, in the limit $\varepsilon \to 0$, these equations are singular, and we study the small $\varepsilon$ behavior using a multiple scales analysis.

2 DYNAMICAL EQUATIONS OF MOTION

2.1 Kinematics

The general formulation for internal as well as external pressure forcing, which we sketch here, mimics that of Dempsey [6], where only external pressure forcing was considered. For clarity and completeness, several results contained in [6] are restated. The mathematical description begins by stationing the undeformed ring parallel to the $XY$ plane and centered at the origin $O$ of the standard $(\hat{i}, \hat{j}, \hat{k})$ basis. In this setting, we utilize the unit vector triad $(\hat{n}_\theta, \hat{t}_\theta, \hat{k})$, where:

\[
\hat{n}_\theta = \cos \theta \hat{i} + \sin \theta \hat{j},
\hat{t}_\theta = \hat{k} \times \hat{n}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j}.
\]

Physically, $(\hat{n}_\theta, \hat{t}_\theta, \hat{k})$ represent the standard unit vector basis $(\hat{i}, \hat{j}, \hat{k})$ rotated through an angle $\theta$ about the $\hat{k}$ direction.

In its natural configuration, the ring is circular with radial wall thickness $h$ and midsurface radius $a$. At $t = 0$ we locate points on a material plane by the position vector:

\[
R(\alpha, \zeta, 0) = (a + \zeta) \hat{n}_\alpha, \quad -h/2 \leq \zeta \leq h/2,
\]

where $\zeta$ is the radial distance from the midsurface and $\alpha$ is the inclination of the material plane from the with respect to the $\hat{i}$ direction. Thus a mass element located at $R(\alpha, \zeta, 0)$ has an initial volume per unit thickness:

\[
dV_0 = (a + \zeta) d\zeta / \alpha.
\]

Using a Lagrangian formulation, this element deforms to position $R(\alpha, \zeta, t)$ at time $t$. In the following, spatial and temporal derivatives are denoted by $(\cdot)' = \partial / \partial \alpha$ and $(\cdot)$ = $\partial / \partial t$ respectively.

2.2 Balance Laws

We consider an element in the reference configuration defined as:

\[
\Omega_0 = \{ R(\alpha, \zeta, 0) : (\alpha, \zeta) \in [\alpha_1, \alpha_2] \times [-h/2, h/2] \},
\]

which, at time $t$, is deformed to:

\[
\Omega_t = \{ R(\alpha, \zeta, t) : R(\alpha, \zeta, 0) \in \Omega_0 \}.
\]

2.2.1 Linear Momentum Balance

Linear momentum balance on $\Omega_t$ yields:

\[
\int_{t_1}^{t_2} \int_{\partial \Omega_t} t_\alpha(\alpha, \zeta, t, \hat{n}) dS_t dt = \int_{\Omega_t} \hat{R}(\alpha, \zeta, t) d\alpha \bigg|_{t_1}^{t_2},
\]

where $t_\alpha(\alpha, \zeta, t, \hat{n})$ is the Cauchy stress vector, and depends on the surface normal $\hat{n}$. The mass element $d\alpha$ can be expressed as $d\alpha = \rho(\alpha, \zeta, t) dV_t$, where $\rho$ is the density. Via conservation of mass, $d\alpha |_{t_1} = d\alpha |_{t_2}$, or specifically:

\[
\rho(\alpha, \zeta, 0) dV_t = \rho(\alpha, \zeta, 0) dV_0, \quad = \rho dV_0,
\]

where the ring is assumed to be initially homogeneous with density $\rho$. The Piola-Kirchhoff identity relates the stress vector $t_\alpha$ to the undeformed state [2], i.e.:

\[
t_\alpha(\alpha, \zeta, t, \hat{n}_0) dS_0 = t_\alpha(\alpha, \zeta, t, \hat{n}) dS_t,
\]

where $t_\alpha$ is the Piola-Kirchhoff stress vector. In terms of the undeformed configuration, linear momentum balance becomes:

\[
\int_{t_1}^{t_2} \int_{\partial \Omega_0} t_\alpha(\alpha, \zeta, t, \hat{n}_0) dS_0 dt = \int_{\Omega_0} \hat{R}(\alpha, \zeta, t) \rho dV_0 \bigg|_{t_1}^{t_2}.
\]

We can decompose the contact force as:

\[
\int_{\partial \Omega_0} t_\alpha dS_0 dt = F(\alpha, t) \bigg|_{\alpha_1}^{\alpha_2} + \int_{\alpha_1}^{\alpha_2} P(\alpha, t) d\alpha.
\]
with:

\[
\begin{align*}
F(\alpha, t) &= \int_{-h/2}^{h/2} \mathbf{t}_p(\alpha, \zeta, t, \hat{\mathbf{n}}_a) \, d\zeta; \\
P(\alpha, t) &= P_+ (\alpha, t) + P_- (\alpha, t), \\
&= \mathbf{t}_p(\alpha, h/2, t, \hat{\mathbf{n}}_a)(a + h/2) + \\
&= \mathbf{t}_p(\alpha, -h/2, t, -\hat{\mathbf{n}}_a)(a - h/2),
\end{align*}
\]

where \( F(\alpha, t) \) is the resultant force exerted by the \([\alpha, \alpha^+]) \) face of the ring on the \((\alpha^-, \alpha]) \) face, and \( P(\alpha, t) \) is the total force (per unit angle) exerted by the pressure on the inner and outer surfaces, \( \zeta = -h/2 \) and \( \zeta = h/2 \), respectively.

We express \( \mathbf{R}(\alpha, \zeta, t) = \mathbf{r}(\alpha, t) + \mathbf{Z}(\alpha, \zeta, t) \), where:

\[
\mathbf{r}(\alpha, t) = \frac{1}{m} \int_{-h/2}^{h/2} \mathbf{R}(\alpha, \zeta, t) (a + \zeta) \rho \, d\zeta,
\]

defines the shell [6], with \( m = \rho a h \), the mass per unit angle. Then \( \mathbf{Z}(\alpha, \zeta, t) \) satisfies the dynamic consistency condition [6, 14]:

\[
\int_{-h/2}^{h/2} \mathbf{Z}(\alpha, \zeta, t) (a + \zeta) \rho \, d\zeta = 0,
\]

and, as a result, the linear momentum becomes:

\[
\int_{\Omega_0} \dot{\mathbf{r}}(\alpha, t) \, dV = \int_{\alpha_1}^{\alpha_2} m \mathbf{r}(\alpha, t) \, d\alpha.
\]

In the undeformed position the shell coincides with the dynamic axis [6]:

\[
\mathbf{r}(\alpha, 0) = \frac{1}{m} \int_{-h/2}^{h/2} \mathbf{R}(\alpha, \zeta, 0) (a + \zeta) \rho \, d\zeta,
\]

\[
= \frac{1}{m} \int_{-h/2}^{h/2} (a + \zeta)^2 \rho \, d\zeta \mathbf{n}_a,
\]

\[
= a \left( 1 + \frac{h^2}{12a^2} \right) \mathbf{n}_a,
\]

\[
= a(1 + \eta) \hat{\mathbf{n}}_a = R \hat{\mathbf{n}}_a.
\]

The dynamic axis initially lies a distance \( a \eta \ll h \) outside the mid-surface.

Returning the above expressions to the balance law:

\[
\int_{t_1}^{t_2} \left\{ \mathbf{F}(\alpha, t) \big|_{t_1}^{t_2} + \int_{\alpha_1}^{\alpha_2} \mathbf{P}(\alpha, t) \, d\alpha \right\} \, dt = \int_{\alpha_1}^{\alpha_2} m \dot{\mathbf{r}}(\alpha, t) \, d\alpha \bigg|_{t_1}^{t_2}.
\]

Provided both \( \mathbf{F}' \) and \( \dot{\mathbf{r}} \) exist, which implies a restriction to classical solutions, we can write the equation for linear momentum balance as:

\[
\mathbf{F}'(\alpha, t) + \mathbf{P}(\alpha, t) = m \dot{\mathbf{r}}(\alpha, t),
\]

2.2.2 Angular Momentum Balance

Angular momentum balance on the annular element \( \Omega_k \) in the \( \mathbf{k} \) direction is expressed as:

\[
\dot{\mathbf{k}} = \mathbf{k} \cdot \left\{ \int_{t_1}^{t_2} \int_{\Omega_0} \mathbf{R}(\alpha, \zeta, t) \times \mathbf{t}_p(\alpha, \zeta, t, \hat{\mathbf{n}}_a) \, dS \, dt \\
= \int_{\Omega_0} \mathbf{R}(\alpha, \zeta, t) \times \dot{\mathbf{R}}(\alpha, \zeta, t) \, dm \right\} _{t_1}^{t_2}.
\]

As with linear momentum balance, we use mass conservation and the second Piola-Kirchhoff tensor to express the angular momentum balance law in a Lagrangian form. Then, via the dynamic consistency condition and restricting to classical solutions, we obtain:

\[
M'(\alpha, t) + \dot{\mathbf{k}} \cdot (\mathbf{r}'(\alpha, t) \times \mathbf{F}(\alpha, t)) + \varrho(\alpha, t) = J \omega(\alpha, t),
\]

2.3 Strains

In the preceding theory, the shell-averaged linear and angular momenta of continuum points initially on the material normal \( \hat{\mathbf{n}}_a \) are \( m \mathbf{r}(\alpha, t) \) and \( J \dot{\beta}(\alpha, t) \dot{\mathbf{k}} \)—the undeformed inerts being consistent with a Lagrangian formulation—where:

\[
\beta(\alpha, t) = \int_0^t \omega(\alpha, t) \, dt.
\]

\( \dot{\mathbf{k}} \) is the counterclockwise moment (at the dynamic axis) exerted by the \([\alpha, \alpha^+]) \) face of the ring on the \((\alpha^-, \alpha]) \) face, \( \varrho(\alpha, t) \) is the counterclockwise moment (per unit angle) due to the external pressure and \( \omega(\alpha, t) \) has units of angular velocity.\(^1\) \( J \) is the polar moment of inertia per unit angle of the undeformed ring about the dynamic axis, and is:

\[
J = \int_{-h/2}^{h/2} (\zeta - a \eta)^2 (a + \zeta) \rho \, d\zeta = \frac{m h^2}{12} (1 - \eta).
\]

\( \dot{\beta}(\alpha, t) \) of continuum points initially on the ortho-normal directions defined as:

\[
\hat{\mathbf{d}}_1 = \hat{\mathbf{n}}_{a+\beta}, \quad \hat{\mathbf{d}}_2 = \hat{\mathbf{k}}, \quad \hat{\mathbf{d}}_3 = \hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2 = \hat{\mathbf{n}}_{a+\beta},
\]

where \( \beta \) defined in Eq. (3), represents the mass-averaged angular rotation of material points initially located on the material normal \( \hat{\mathbf{n}}_a \). We emphasize that \( \hat{\mathbf{n}}_{a+\beta} \) is not necessarily in the direction of the shell normal.

\(^1\)Although we use the theory of beamshells developed by Libai and Simmonds [14], Eqs. (1) and (2) can be obtained using a special Cosserat theory for the ring [2, Ch. VIII], with the ortho-normal directions defined as:

\[
\hat{\mathbf{d}}_1 = \hat{\mathbf{n}}_{a+\beta}, \quad \hat{\mathbf{d}}_2 = \hat{\mathbf{k}}, \quad \hat{\mathbf{d}}_3 = \hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2 = \hat{\mathbf{n}}_{a+\beta},
\]
Note that \( r(\alpha, t) \) is not necessarily in the \( \hat{n}_{\alpha + \beta} \) direction. If the shell position vector \( r(\alpha, t) \) and inertial rotation \( \beta(\alpha, t) \) are chosen as kinematic variables, the shear, extensional and bending strains associated with the shell are [2, 6, 14]:

\[
\Gamma(\alpha, t) = \frac{r'(\alpha, t) \cdot \hat{n}_{\alpha + \beta}}{R},
\]

\[
E(\alpha, t) = \frac{r'(\alpha, t) \cdot \hat{t}_{\alpha + \beta}}{R} - 1,
\]

\[
B(\alpha, t) = \frac{\beta(\alpha, t)}{R},
\]

respectively. Clearly \( \dot{\beta}(\alpha, t) = \omega(\alpha, t). \) Since \( \Gamma(\alpha, t) \) and \( E(\alpha, t) \) are the unique scalars that enable \( r'(\alpha, t) \) to be written in the form:

\[
r'(\alpha, t) = R \left( \Gamma(\alpha, t) \hat{n}_{\alpha + \beta} + (1 + E(\alpha, t)) \hat{t}_{\alpha + \beta} \right),
\]

the corresponding shear and extensional continuum strains are inferred to be:

\[
\gamma(\alpha, \zeta, t) = \frac{R' \left( \Gamma(\alpha, \zeta, t) \cdot \hat{n}_{\alpha + \beta} \right)}{a + \zeta},
\]

\[
\gamma(\alpha, \zeta, t) = \frac{R \Gamma(\alpha, t) + Z(\alpha, \zeta, t) \cdot \hat{n}_{\alpha + \beta}}{a + \zeta},
\]

\[
\xi(\alpha, \zeta, t) = \frac{R(1 + E(\alpha, t)) G(\alpha, \zeta, t) \cdot \hat{t}_{\alpha + \beta}}{a + \zeta} - 1,
\]

so that:

\[
R'(\alpha, \zeta, t) = (a + \zeta) \left( \gamma(\alpha, \zeta, t) \hat{n}_{\alpha + \beta} + (1 + \xi(\alpha, \zeta, t)) \hat{t}_{\alpha + \beta} \right).
\]

### 2.4 Constitutive Relations

We make the following assumptions concerning the behavior of the ring:

(i) material planes remain plane and undeformed;

(ii) the material of the ring is elastic and obeys the familiar Hooke's law.

Note that material planes remain normal to the shell only if shear deformation is suppressed—(i) is a variant of the classical Kirchhoff-Love hypothesis [1]. We consider these assumptions in turn, see Dempsey [6] for further details.

(i) If material planes remain plane and undeformed, the shell at time \( t \) is simply the deformed image of the dynamic axis, i.e.:

\[
r(\alpha, t) = R(\alpha, \alpha \eta, t).
\]

This implies that the Cauchy and Piola-Kirchhoff stress vectors, \( t(\alpha, \zeta, t, \hat{t}_{\alpha + \beta}) \) and \( \hat{t}_p(\alpha, \zeta, t, \hat{t}_{\alpha + \beta}) \) on the material planes are equal in the deformed and undeformed configurations. Thus we have the following:

\[
Z(\alpha, \zeta, t) = (\zeta - \alpha \eta) \hat{n}_{\alpha + \beta},
\]

\[
t_p(\alpha, \zeta, t, \hat{t}_{\alpha + \beta}) = t(\alpha, \zeta, t, \hat{t}_{\alpha + \beta}),
\]

\[
\tau(\alpha, \zeta, t) \hat{n}_{\alpha + \beta} + \sigma(\alpha, \zeta, t) \hat{t}_{\alpha + \beta},
\]

where \( \tau(\alpha, \zeta, t) \) and \( \sigma(\alpha, \zeta, t) \) are the continuum shear and extensional stresses, respectively. As a result, the continuum strains reduce to:

\[
\gamma(\alpha, \zeta, t) = \frac{R \Gamma(\alpha, t)}{a + \zeta},
\]

\[
\xi(\alpha, \zeta, t) = \frac{R E(\alpha, t) + (\zeta - \alpha \eta) \beta'}{a + \zeta}.
\]

In particular, \( \gamma(\alpha, \alpha \eta, t) = \Gamma(\alpha, t) \) and \( \xi(\alpha, \alpha \eta, t) = E(\alpha, t) \) —the shell strains \( \Gamma(\alpha, t) \) and \( E(\alpha, t) \) represent the continuum strains on the dynamic axis. In addition, the deformed position \( R(\alpha, \zeta, t) \) is:

\[
R(\alpha, \zeta, t) = r(\alpha, t) + Z(\alpha, \zeta, t),
\]

\[
= r(\alpha, t) + (\zeta - \alpha \eta) \hat{n}_{\alpha + \beta}.
\]

(ii) Within the confines of a linearly elastic material, the continuum shear and extensional stresses vary linearly with their corresponding strains:

\[
\tau(\alpha, \zeta, t) = \frac{kE}{2(1 + \nu)} \gamma(\alpha, \zeta, t),
\]

\[
\sigma(\alpha, \zeta, t) = \frac{E}{1 - \nu^2} \xi(\alpha, \zeta, t).
\]

\( E \) and \( \nu \) are Young’s modulus and Poisson’s ratio, respectively, while \( k \) is an anticipatory shear correction factor. The latter is traditionally arrived at by actual through-the-thickness integration of the equilibrium equations for known solutions in the expected spectrum of dynamic motions [5]. If the material force \( F(\alpha, t) \) is expressed as:

\[
F(\alpha, t) = Q(\alpha, t) \hat{n}_{\alpha + \beta} + N(\alpha, t) \hat{t}_{\alpha + \beta},
\]

then:

\[
Q(\alpha, t) = \int_{-h/2}^{h/2} \tau(\alpha, \zeta, t) d\zeta,
\]

\[
= \frac{kE}{2(1 + \nu)} \frac{\Gamma}{l},
\]

\[
N(\alpha, t) = \int_{-h/2}^{h/2} \sigma(\alpha, \zeta, t) d\zeta,
\]

\[
= \frac{E}{1 - \nu^2} \frac{E}{l} - (R - l) \beta',
\]

\[
M(\alpha, t) = \frac{E}{1 - \nu^2} \left( al - 2RL + R^2 \right) \beta - (R - l) \beta',
\]
with:
\[
l = h \left( \ln \left| \frac{a + h/2}{a - h/2} \right| \right)^{-1} = a(1 - \eta + \mathcal{O}(\eta^2)),
\]
which represents the radius of the neutral axis.

### 2.5 Pressure Loading

For spatially uniform hydrodynamic pressure, the Cauchy stress vector on the internal and external surfaces can be expressed as:
\[
\begin{align*}
t_i(\alpha, -h/2, t, \hat{n}_-) &= -p_i(t)\hat{n}_-(\alpha, t), \\
t_e(\alpha, h/2, t, \hat{n}_+) &= -p_e(t)\hat{n}_+(\alpha, t),
\end{align*}
\]
where \( \hat{n}_- \) and \( \hat{n}_+ \) respectively are the outward unit normal on the internal and external surfaces of the ring. The corresponding Pissa-Kirchhoff pressures are:
\[
\begin{align*}
t_\alpha(\alpha, -h/2, t, -\hat{n}_+) &= -S_-(\alpha, t)p_\alpha(t)\hat{n}_-(\alpha, t), \\
t_\alpha(\alpha, h/2, t, \hat{n}_+) &= -S_+(\alpha, t)p_\alpha(t)\hat{n}_+(\alpha, t),
\end{align*}
\]
where \( S_\pm(\alpha, t) \) represents the stretch factor on the corresponding surface, and is given as:
\[
S_\pm(\alpha, t) = \sqrt{\gamma_\pm^2 + (1 + \xi_\pm)^2}.
\]
Here \( \gamma_\pm = (\alpha, \pm h/2, t) \) and \( \xi_\pm = \xi(\alpha, \pm h/2, t) \). Finally, \( \mathbf{P}(\alpha, t) \) and \( \varrho(\alpha, t) \) become:
\[
\begin{align*}
\mathbf{P}(\alpha, t) &= \mathbf{P}_+(\alpha, t) + \mathbf{P}_-(\alpha, t), \\
\varrho(\alpha, t) &= \frac{\mathbf{R}(\alpha, t) p_\alpha(t) + p_\epsilon(t)}{1 + \xi(\alpha, t)} \hat{n}_{\alpha + \beta} - \left\{ \frac{\mathbf{R}(\alpha, t) p_\alpha(t) + p_\epsilon(t)}{1 + \xi(\alpha, t)} \right\} \hat{\ell}_{\alpha + \beta},
\end{align*}
\]

### 2.6 Nondimensional Equations of Motion

Time is nondimensionalized according to:
\[
t \to \sqrt{\frac{(1 - \nu^2) p_0}{E h}} t,
\]
so that the frequency of free breathing vibration is unity [6], and the pressure is scaled as:
\[
p(t) \to \frac{(1 - \nu^2) l}{E h} p(t).
\]
The ring and its deformation are scaled by the dynamic axis radius \( R \), so that the undeformed ring is the unit circle \( r(\alpha, 0) = \hat{n}_\alpha \). Let:
\[
\mu = \frac{k(1 - \nu)}{2}, \quad \varepsilon^2 = \frac{J}{mR^2} = \eta \left( \frac{1 - \eta}{(1 + \eta)^2} \right).
\]
Then, under the assumptions outlined above—a relaxed Kirchhoff-Love hypothesis, hydrodynamic forcing, and a linearly elastic material—the nondimensional equations of motion are:
\[
\begin{align*}
\ddot{r}(\alpha, t) &= F'(\alpha, t) + P(\alpha, t), \\
\lambda(\alpha, t) &= M'(\alpha, t) + \dot{K} \cdot \left( \frac{r'(\alpha, t) \times F(\alpha, t)}{g(\alpha, t)} \right),
\end{align*}
\]
with:
\[
\begin{align*}
F(\alpha, t) &= Q(\alpha, t)\hat{n}_{\alpha + \beta} + N(\alpha, t)\hat{\ell}_{\alpha + \beta}, \\
P(\alpha, t) &= P_n(\alpha, t)\hat{n}_{\alpha + \beta} + P_i(\alpha, t)\hat{\ell}_{\alpha + \beta},
\end{align*}
\]
where:
\[
\begin{align*}
Q(\alpha, t) &= \mu \Gamma, \\
N(\alpha, t) &= E \left( 1 - (1 - \lambda)\beta' \right), \\
M(\alpha, t) &= \left( 1 - 2\lambda + \frac{\lambda}{1 + \eta} \right) \beta' - (1 - \lambda)\varepsilon, \\
P_n(\alpha, t) &= \left( E + \frac{1 - \eta\beta'}{1 + \eta} \right) (p_\alpha(t) - p_\epsilon(t)) - \frac{\delta}{2} (1 + \beta) (p_\alpha(t) + p_\epsilon(t)), \\
P_i(\alpha, t) &= -\Gamma (p_\alpha(t) - p_\epsilon(t)), \\
\varepsilon(\alpha, t) &= \Gamma \left( \frac{\delta}{2} (p_\alpha(t) + p_\epsilon(t)) + \frac{\eta}{1 + \eta} (p_\alpha(t) - p_\epsilon(t)) \right),
\end{align*}
\]
and finally:
\[
\begin{align*}
\Gamma(\alpha, t) &= r'(\alpha, t) \cdot \hat{n}_{\alpha + \beta}, \\
\varepsilon(\alpha, t) &= r'(\alpha, t) \cdot \hat{\ell}_{\alpha + \beta} - 1,
\end{align*}
\]
where we identify the (nondimensional) ring thickness \( \delta \) and neutral axis radius \( \lambda \) as:
\[
\delta = \frac{h}{R} = \frac{\sqrt{2\eta}}{1 + \eta}, \quad \lambda = \frac{l}{R} = \delta \left( \ln \left| \frac{2 + \delta(1 + \eta)}{2 - \delta(1 + \eta)} \right| \right)^{-1}.
\]
Of the nondimensional parameters \( \varepsilon, \delta, \lambda, \) and \( \eta \), only one is independent. We note that the unforced variants of the coordinate-free Eqs. (5) are identical in form to those originally obtained by Simmonds [23].
the motion of the symmetric elastic ring are

\[ \ddot{r} - (1 + r) \frac{d}{dt} \dot{\phi}^2 = \cos \gamma - (1 + r)(\cos^2 \gamma + \mu \sin^2 \gamma) - (P_0(t)(1 + r) + P_1(t) \cos \gamma), \quad (7a) \]

\[ (1 + r) \ddot{\phi} + 2 \dot{r} \dot{\phi} = \sin \gamma (1 - (1 + r)(1 - \mu) \cos \gamma - P_1(t)), \quad (7b) \]

\[ \varepsilon^2 \ddot{\beta} = -(1 + r) \sin \gamma (1 - (1 + r)(1 - \mu) \cos \gamma - P_1(t)), \quad (7c) \]

where:

\[ \gamma = \beta - \phi, \]

\[ P_0(t) = p_e(t) - p_i(t), \]

\[ P_1(t) = \frac{\sqrt{\varepsilon^2 - \eta}}{1 + \eta} p_e(t) + \frac{\sqrt{\varepsilon^2 + \eta}}{1 + \eta} p_i(t). \]

\( P_0(t) \) represents the difference between the external and internal hydrostatic loading, while \( P_1(t) \) is a weighted average of the pressures. Observe that the nondimensional equations depend only on two parameters: \( \eta = \frac{k^2}{12a^2} \ll 1 \), which characterizes the ratio of the thickness of the ring to its radius, and \( \mu = k(1 - \nu)/2 \), which describes the material properties. In addition, note that \( \eta \) only appears in the nondimensional equations through the forcing term \( P_1(t) \) and \( \varepsilon \), the nondimensional mass radius of gyration.

Equations (7b) and (7c) can be combined to yield \( (1 + r)^2 \ddot{\phi} + 2(1 + r) \ddot{\phi} + \varepsilon^2 \ddot{\beta} = 0 \), which we recognize as a statement of conservation of angular momentum for the system:

\[ (1 + r)^2 \frac{d^2}{dt^2} \phi + \varepsilon^2 \beta = \kappa = \text{constant}. \quad (8) \]

### 3 SYMMETRIC RESPONSE

#### 3.1 Routh’s Procedure

We note that Eqs. (7) can be derived from Lagrange’s equations:

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \]

with the Lagrangian \( L = T - V \), given as:

\[ T = \frac{1}{2} \left( u^2 + (u \dot{\phi})^2 + (\varepsilon \dot{\beta})^2 \right), \]

\[ V = (1 - P_1(t)) u \cos \gamma - \frac{u^2}{2} \left( (1 - \mu) \cos^2 \gamma + (\mu + P_0(t)) \right), \]

with \( u = 1 + r \). To the Lagrangian, we make the transformation \( \beta = \phi + \gamma \), to eliminate \( \beta \) from \( T \), the kinetic energy. Within this framework, \( \phi \) is an ignorable (or cyclic) coordinate, which implies that its conjugate momentum is conserved. We use a procedure originally devised by Routh [21] to eliminate \( \phi \), which involves a transformation from \( (\phi, \beta) \) to \( (\phi, \kappa) \), where \( \kappa \) is the momentum conjugate to \( \phi \), defined as:

\[ \kappa = \frac{\partial L}{\partial \dot{\phi}} = (u^2 + \varepsilon^2) \dot{\beta} + \varepsilon^2 \dot{\gamma}. \]
The choice of nomenclature $\kappa$ for the conjugate momentum is intentional—the momentum conjugate to $\phi$ represents the angular momentum of the system, and, as will we verify below, is conserved. Finally, using a Legendre transformation, we define the Routhian $\mathcal{R}(u, \dot{u}, \gamma, \dot{\gamma}, \kappa)$:

$$\mathcal{R} = \kappa \dot{\phi} - \mathcal{L} = \frac{(\kappa - \varepsilon^2 \gamma^2)^2}{2(u^2 + \varepsilon^2)} - \frac{1}{2}(\dot{u}^2 + (\varepsilon \dot{\gamma})^2) + V(u, \gamma),$$

and, a differential of $\mathcal{R}$ yields:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{R}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{R}}{\partial q_i} = 0, \quad q_i = (u, \gamma),$$

$$\dot{\phi} = \frac{\partial R}{\partial u},$$

$$\kappa = -\frac{\partial R}{\partial \phi} = 0.$$

As promised, the conjugate momentum $\kappa$ is constant, and $\phi$, by virtue of its independence from the Routhian, decouples from the remaining equations on $u$ and $\gamma$, which are described by Lagrange's equations with $\mathcal{R}$ as the Lagrangian. The equations of motion on $u$ and $\gamma$ are independent of the cyclic coordinate $\phi$, and can be expressed as:

$$\ddot{u} = u \frac{(\kappa - \varepsilon^2 \gamma^2)^2}{(u^2 + \varepsilon^2)} + \left(1 - P_1(t)\right) \cos \gamma - u \left(1 - \mu \right) \cos^2 \gamma + (\mu + P_0(t)), \quad (9a)$$

$$\varepsilon^2 \ddot{\gamma} = \varepsilon^2 \frac{2(\kappa - \varepsilon^2 \gamma^2)}{u^2 + \varepsilon^2} \frac{\dot{u}}{u} - \sin \gamma \left(u + \varepsilon^2 \frac{\dot{\gamma}}{u}\right) \times \left(1 - P_1(t) - (1 - \mu)u \cos \gamma\right). \quad (9b)$$

Thus, using Routh’s procedure we have eliminated one degree-of-freedom corresponding to the ignorable coordinate $\phi$, although the dynamical behavior of the system certainly depends on $\kappa$, which is a function of the initial state. We note that $\gamma \equiv 0$ is, in general, not a solution to these equations.

Through the transformation $\dot{u} = v, \dot{\gamma} = \psi/\kappa$, we write these equations as a first order system. In Figure 2 we present sample numerical integrations for $\kappa = 1.5$ in the absence of pressure loads. We use a fourth-order Runge-Kutta method and, because of the singular nature of the system, a step size $\Delta t = 10^{-5}$. In Figure 2a, we choose several initial conditions on $u(0)$, while fixing $v(0) = 0, \gamma(0) = 0, \text{and } \psi(0) = 0$. However, in Figure 2b, a single trajectory with $u(0) = 1$ is shown. We note that when projected onto $(\gamma, \psi)$, the solution appears to possess two different behaviors—oscillation around $\gamma = 0$ and oscillation about some (slowly-varying) nonzero value of $\gamma$.

### 3.2 Unperturbed System

In the limit $\varepsilon \to 0$ the moment of inertia of material planes around the dynamic axis vanishes—the ratio of the ring thickness to its undeformed radius is zero. In this limit $\gamma = 0$ and the term $P_1(t)$ vanishes, so that Eqs. (9) reduce to:

$$\ddot{u} = \kappa^2 \frac{u^{3}}{u^3} - u(1 + P_0(t)) + 1. \quad (10)$$

This is simply a forced single degree-of-freedom system, and for general time dependent forcing, allows for complicated dynamical behavior, including parametric resonances and chaotic behavior. However, we consider the applied pressure loading to be constant, so that $P_0(t) = p_0$, and phase space is filled with periodic orbits. A representative phase portrait of the unperturbed system is shown in Figure 3. In the remaining work, we will study how this simple picture changes for $\varepsilon \neq 0$.

#### 3.3 Relative Equilibria

We seek to characterize a particular class of solutions, known as relative equilibria, subject to constant internal and external pressure, so that $P_0(t) = p_0$ and $P_1(t) = p_1$. Physically, relative equilibria correspond to uniformly rotating solutions, so that $\dot{u} = 0$ and $\dot{\gamma} = 0$. As a result, the term $\frac{d}{dt} \left( \frac{\partial \mathcal{R}}{\partial \dot{\gamma}} \right)$ vanishes, so that Lagrange's equations provide the algebraic conditions at equilibrium:

$$0 = \frac{\partial \mathcal{R}}{\partial u} \bigg|_{u_{eq}, \gamma_{eq}}, \quad 0 = \frac{\partial \mathcal{R}}{\partial \gamma} \bigg|_{u_{eq}, \gamma_{eq}},$$

which reduce to:

$$0 = \frac{\kappa^2 u_{eq}}{(u_{eq}^2 + \varepsilon^2)} + (1 - p_1) \cos \gamma_{eq} - u_{eq} \left((1 - \mu) \cos^2 \gamma_{eq} + (\mu + p_0)\right),$$

$$0 = -\sin \gamma_{eq} (u_{eq}^2 + \varepsilon^2) \times \left((1 - p_1) - (1 - \mu)u_{eq} \cos \gamma_{eq}\right).$$
There exists a branch of relative equilibria which satisfy:

\[ \gamma_{eq} = 0, \quad (11a) \]

\[ \frac{K^2 u_{eq}}{(u_{eq}^2 + \varepsilon^2)^2} + (1 - p_1) - u_{eq}(1 + p_0) = 0, \quad (11b) \]

which, with \( \dot{\phi}_{eq} = \omega \), implies:

\[ u_{eq} = \frac{(1 - p_1)}{(1 + p_0)} - \omega^2. \]

In addition, there exists a second nontrivial branch of relative equilibria which are determined by:

\[ \cos \gamma_{eq} = \frac{1 - p_1}{(1 - \mu)u_{eq}}, \quad (12a) \]

\[ u_{eq} = \sqrt{\frac{K}{\sqrt{\mu + p_0} - \varepsilon^2}}. \quad (12b) \]

Clearly, solutions on this nontrivial branch exist only if \( |\cos \gamma_{eq}| \leq 1 \), which implies that:

\[ |u_{eq}| \geq \frac{1 - p_1}{1 - \mu} \equiv u_{cr}. \]

Again, in terms of \( \dot{\phi}_{eq} = \omega \), this implies that:

\[ \omega = \sqrt{\mu + p_0} \equiv \omega_{cr}, \]

so that on this branch of equilibria the angular rotation rate of the elastica is independent of the angular momentum \( \kappa \) of the system. This is reminiscent of a non-generic family of solutions identified in Nayfeh et al. [18]. On this nontrivial branch, the angular momentum satisfies:

\[ \kappa \geq \left[ \frac{(1 - p_1)^2}{(1 - \mu)^2 + \varepsilon^2} \right] \sqrt{\mu + p_0} \equiv \kappa_{cr}. \]

We briefly discuss the physical implications of these relative equilibria. \( \gamma_{eq} = 0 \) implies that in the relative equilibrium configuration, material planes are aligned normal to the dynamic axis—the shear strain vanishes. However, for \( \gamma_{eq} \neq 0 \), the material planes are misaligned with the dynamic axis normal and in this relative equilibrium configuration there exists a nonzero shear strain in the ring. We note that these equations also admit an equilibrium with \( \gamma_{eq} = \pi \), which is physically unrealistic and hence will not be considered.

Within this restricted space of \( SO(2) \) symmetric solutions, the stability of the relative equilibria is relatively straightforward, although the calculations are cumbersome. If the second variation of the Routhian in the neighborhood of the relative equilibria is convex, the equilibrium is linearly stable. We find that for \( \kappa < \kappa_{cr} \) the equilibrium with \( \gamma_{eq} = 0 \) is linearly stable. However, as \( \kappa \) is increased through this bifurcation value, there exists a pitchfork bifurcation and the unheared state becomes unstable. The bifurcating branch of solutions, which are described in Eqs. (12) is linearly stable. We note that the stability of an equilibrium in Eqs. (9) implies that the corresponding equilibria in the physical system possesses orbital Lyapunov stability.

### 3.4 Poincaré Sections

Numerically we study the dynamical behavior of this system through Poincaré sections of the flow. We fix the initial conditions on \( u \) and \( v \) in their undeformed position, i.e. \((u(0), v(0)) = (1, 0)\) and mark the projection onto \((\gamma, \psi)\) when the resulting trajectory crosses \( v = 0 \) with \( \dot{v} < 0 \). If \( \kappa \) is sufficiently small and the initial conditions are chosen so that the amplitude of \( u \) remains below \( u_{cr} \), the resulting Poincaré section shows regular behavior, as seen in Figure 4a. However, if the amplitude of the radial coordinate \( u \) is sufficiently large, so that \( u > u_{cr} \) during an interval of the motion, the Poincaré section no longer shows regular structure. Indeed, as the angular momentum is increased, we see a small region of irregular motion near the origin in Figure 4b. In Figures 4c-d, in which the angular momentum is further increased, the region of irregular motion is significant.

### 3.5 Singly Perturbed System

Because of the presence of terms \( \sim \frac{1}{\varepsilon} \), in the limit \( \varepsilon \to 0 \) these equations are singular. In fact, such terms indicate that the motion evolves on two time-scales—one of \( O(1) \) and a second, fast time-scale of \( O(\frac{1}{\varepsilon}) \). Written as a first-
order system, the equations become:

\[
\begin{align*}
\frac{du}{dt} &= v, \\
\frac{dv}{dt} &= \frac{u(k - \varepsilon\psi)^2}{(u^2 + \varepsilon^2)^2} + (1 - p_1) \cos \gamma - u \left(1 - (1 - \mu) \cos^2 \gamma + (\mu + p_0)\right), \\
\frac{d\gamma}{dt} &= \psi, \\
\frac{d\psi}{dt} &= -u \sin \gamma \left((1 - p_1) - (1 - \mu) u \cos \gamma\right) + \varepsilon^2 \left(\frac{2u(k - \varepsilon\psi)}{(u^2 + \varepsilon^2)} - \sin \gamma \left((1 - p_1) - (1 - \mu) u \cos \gamma\right)\right).
\end{align*}
\]

To incorporate the fast time-scale, we introduce the new independent variable:

\[
\tau = \frac{t}{\varepsilon^\frac{1}{2}}
\]

so that the total derivative with respect to time becomes:

\[
\frac{d}{dt} = \frac{\partial}{\partial \tau} + \varepsilon^{-1} \frac{\partial}{\partial \tau}
\]

We use a regular expansion of the state variables in \(\varepsilon\):

\[
\begin{align*}
 u(t, \tau) &= u_0(t, \tau) + \varepsilon u_1(t, \tau) + \cdots, \\
 v(t, \tau) &= v_0(t, \tau) + \varepsilon v_1(t, \tau) + \cdots, \\
 \gamma(t, \tau) &= \gamma_0(t, \tau) + \varepsilon \gamma_1(t, \tau) + \cdots, \\
 \psi(t, \tau) &= \psi_0(t, \tau) + \varepsilon \psi_1(t, \tau) + \cdots,
\end{align*}
\]

and, collecting terms at each order in \(\varepsilon\):

\[
\mathcal{O}(\varepsilon^{-1}): \\
\frac{du_0}{d\tau} &= 0, \\
\frac{dv_0}{d\tau} &= 0, \\
\frac{d\gamma_0}{d\tau} &= \psi_0, \\
\frac{d\psi_0}{d\tau} &= -u_0 \sin \gamma_0 \times (1 - p_1) - (1 - \mu) u_0 \cos \gamma_0, \\
\mathcal{O}(\varepsilon^0): \\
\frac{du_0 + \varepsilon u_1}{d\tau} &= v_0, \\
\frac{dv_0 + \varepsilon v_1}{d\tau} &= \frac{k^2}{u_0^3} + (1 - p_1) \cos \gamma_0 - u_0 \left((1 - \mu) \cos^2 \gamma_0 + (\mu - p_0)\right).
\]

We note that in this system we have not identified the \(\mathcal{O}(\varepsilon^0)\) evolution equations on \(\gamma\) and \(\psi\). These equations shall not play a role in our resulting analysis. We find that only the lowest order terms \(\gamma_0\) and \(\psi_0\) effect the dynamical behaviour of \(u_0\) and \(v_0\).

Examination of these equations at \(\mathcal{O}(\varepsilon^{-1})\) indicates that \(u_0\) and \(v_0\) do not evolve on the fast time-scale, i.e., they remain constant as \(\tau\) varies. Consequently, on this time-scale the evolutions of \(\gamma_0\) and \(\psi_0\) are described by a simple one degree-of-freedom planar vector field, possessing either two or four equilibria, with \(\psi_0 = 0\), and:

\[
\sin \gamma_{eq} = 0, \quad \cos \gamma_{eq} = \frac{(1-p_1)}{(1-\mu)u_0}, \quad \text{for all } u_0, \quad u_0 > u_{cr}, \quad \text{with } u_{cr} = \frac{(1-p_1)}{(1-\mu)}.
\]

Each of these conditions implies two equilibria. We again find that \(\gamma_{eq} = \pi\) is physically unrealistic and always unstable—thus we only consider \(\gamma_{eq} = 0\) and the two nontrivial equilibria. The existence of the nontrivial equilibria depends on \(u_0\)—the instantaneous value of the radius of the ring. For \(u_0\) less than \(u_{cr}\), defined above, the origin of this system is the only equilibrium point and is stable. However, as \(u_0\) increases through \(u_{cr}\), a pitchfork bifurcation occurs at the origin; the origin becomes unstable and two additional stable equilibria are born, summarized in Figure 5. We note that \(u_{cr}\), which in the previous section determined the existence of relative equilibria in the full equations, also characterizes the pitchfork bifurcation in the fast time-scale. In Figure 6 we present qualitative descriptions of the phase portrait on \(\gamma_0\) and \(\psi_0\) as \(u(t)\) varies.
Returning to the $O(1)$ time-scale, we can solve for $u_1(t, \tau)$ and $v_1(t, \tau)$:

\[
\begin{align*}
    u_1(t, \tau) &= \left\{ v_0(t) - \frac{\partial u_0}{\partial t} \right\} \tau + \tilde{u}_1(t), \\
    v_1(t, \tau) &= \left\{ \frac{\kappa}{v_0(t)^2} - (\mu + p_0)u_0(t) - \frac{\partial v_0}{\partial t} \right\} \tau + \\
    &\left(1 - p_1\right) \int_0^\tau \cos \gamma_0(t, \tau) d\tau - \\
    &\left(1 - \mu\right) u_0(t) \int_0^\tau \cos^2 \gamma_0(t, \tau) d\tau + \tilde{v}_1(t).
\end{align*}
\]

We require that the resulting solution is uniformly valid in time, which implies that we must remove the secular terms in the above expressions for $u_1(t, \tau)$ and $v_1(t, \tau)$ [12]. This provides the following evolution equations on $u_0(t)$ and $v_0(t)$:

\[
\begin{align*}
    \frac{\partial u_0}{\partial t} &= v_0, \\
    \frac{\partial v_0}{\partial t} &= \frac{\kappa}{v_0^2} - (\mu + p_0)u_0 + \\
    &\lim_{t \to \infty} \frac{1}{\tau} \left\{ (1 - p_1) \int_0^\tau \cos \gamma_0(t, \tau) d\tau - \\
    &\left(1 - \mu\right) u_0(t) \int_0^\tau \cos^2 \gamma_0(t, \tau) d\tau \right\},
\end{align*}
\]

where $\gamma_0(t, \tau)$ is determined from the $O(\varepsilon^{-1})$ system.

Examination of these equations indicates that, to this order of the approximation, only the average values of $\cos \gamma_0$ and $\cos^2 \gamma_0$ on the fast time-scale $\tau$ affect the evolution of $(u_0, v_0)$ on the slow time-scale $t$. Considering the $\cos \gamma_0$ term, we define this mean as:

\[
\lim_{t \to \infty} \frac{1}{\tau} \left\{ \int_0^\tau \cos \gamma_0(t, \tau) d\tau \right\} \equiv \cos \vec{\gamma},
\]

where $\gamma_0(t, \tau)$ is a solution on the fast time-scale. Therefore, considered as a function of the trajectory of $\gamma_0(t, \tau)$, $\vec{\gamma}$ varies as this trajectory changes. In Figure (7) we show $\vec{\gamma}$ as obtained from direct integration of the $O(\varepsilon^{-1})$ equations. This figure corresponds to $u(t) > u_{cr}$, and is generated by fixing the initial condition $\gamma_0(0) = \gamma_{eq}$ and allowing $\psi_0(0)$ to vary. Thus we are able to sample all trajectories in the $O(\varepsilon^{-1})$ system which oscillate around the stable equilibria $(\gamma_{eq}, 0)$. As $\psi_0(0) \to 0$ the corresponding trajectory approaches the equilibrium and we find that the mean inclination approaches $\gamma_{eq}$. In the figure, the cusp which occurs near $\psi_0(0) \approx 0.35$ corresponds to a homoclinic orbit of the origin, on which the mean inclination is zero (trajectories on the homoclinic orbit approach the origin as $t \to \pm \infty$).

At $O(\varepsilon^{-1})$ the trajectories of $(\gamma_0, \psi_0)$ oscillate around the stable equilibria, and, as a first approximation we assume $\vec{\gamma} = \gamma_{eq}$ corresponding to the stable equilibria on the fast time-scale (see Figure 6). Recall that as $u_0$ evolves, the value of $\gamma_{eq}$ corresponds to the stable equilibrium on the fast time-scale and undergoes a pitchfork bifurcation as $u_0$
passes through \( u_c \). Also, these two non-trivial stable equilibria have identical effects on \( u_0 \) and \( v_0 \), thus we do not distinguish between the two. A similar analysis provides the approximation:

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \left\{ \int_0^\tau \cos^2 \gamma_0(t, \tau) \, dt \right\} \sim \cos^2 \gamma_{eq}(t),
\]

Equations (13) are of the form:

\[
\begin{align*}
\dot{x} &= f(x, y, \varepsilon), \\
\dot{y} &= g(x, y, \varepsilon),
\end{align*}
\]

where \( x = (u, v) \) and \( y = (\gamma, \psi) \). Here \( g = 0 \) is often referred to as the slow manifold and, in the \( \mathcal{O}(\varepsilon^{-1}) \) equations corresponds to equilibria \( y_{eq} \) (which depend on \( x \)). If these equilibria are asymptotically stable the slow manifold is attractive, and an arbitrary initial condition moves quickly ("jumps") to a point on the slow manifold. The motion then proceeds within the slow manifold according to \( \dot{x} = f(x, y_{eq}) \). This approximation represents an "outer" solution in the language of matched asymptotics, the "inner" solution being given by the description of the initial jump.

However, the equilibria of the \( \mathcal{O}(\varepsilon^{-1}) \) equations are not asymptotically stable and hence the slow manifold is not attractive. Rather, this system is conservative and the trajectories oscillate in the neighborhood of neutrally stable equilibria. Nonetheless, as shown above, the outer solution is only affected by the mean of \( \cos \gamma_0 \) and \( \cos^2 \gamma_0 \). Thus, our above approximation is equivalent to again collapsing the dynamics of \( y \) onto the slow manifold.

**FIGURE 7:** \( \tau \) as defined in Eq. 14; \( \mu = 0.36, p_0 = 0, u_0 = 2.0 \). The dotted line marks the stable equilibrium inclination.

**FIGURE 8:** The lowest order system; \( \varepsilon = 0, \mu = 0.36, p_0 = 0, p_1 = 0 \). For these parameters, \( \kappa_{cr} = 1.47 \). The dashed line denotes the location of \( u_c \). We note that for the initial condition \( u(0) = 1, v(0) = 0 \), for \( \kappa = 0.5 \) the resulting trajectory lies below \( u_c \) for all time, as seen in (a), while in (b-d), with these initial conditions \( u > u_c \) for an interval of the motion (cf. Figure 4).

Finally, to lowest order, the equations governing \( u_0 \) and \( v_0 \) are determined to be:

\[
\begin{align*}
\frac{du_0}{dt} &= v_0, \\
\frac{dv_0}{dt} &= \frac{\kappa^2}{w_0^2} (1 - p_1) \cos \gamma_{eq} - u_0 \left( (1 - \mu) \cos^2 \gamma_{eq} + (\mu - p_0) \right),
\end{align*}
\]

where:

\[
\cos \gamma_{eq} = \begin{cases} 
1, & u_0 \leq u_c, \\
\frac{(1 - p_1)}{(1 - \mu)u_0}, & u_0 > u_c,
\end{cases}
\]

which results in a piecewise continuous vector field, approximating the dynamical behavior of the original system. We note that this lowest order system no longer possesses motion on the fast time-scale \( \tau \). Rather, Eqs. (15) contain no terms evolving on the fast time-scale, yet describe the behavior of Eqs. (9) for \( \varepsilon \ll 1 \). As \( \kappa \) is varied, the resulting phase portrait of this lowest order model is shown in Figure 8.

We note Eqs. (15) are symmetric about the \( u \) axis, so that the lowest order system possesses the symmetry \( (u, v, t) \rightarrow (u, -v, -t) \), and we find that phase space is filled with periodic orbits for \( u > 0 \). Although, this system is independent of \( \varepsilon \), the effect of the nontrivial state \( \gamma_{eq} \neq 0 \)
is noticeable when compared to the unperturbed system (cf. Figure 3). If the radial coordinate remains small, so that $u(t) < \frac{\lambda - \beta_0}{1 - \mu}$, the behavior of the lowest order system is identical to that of the unperturbed system. However, as the magnitude of $u$ increases, the effect of $\gamma \neq 0$ becomes noticeable. We note the agreement between the motions of the lowest order system, in particular that shown in Figure 8d, and the perturbed behavior, shown in Figure 2a, which supports the approximation made on the dynamical behavior of $\gamma_0$. This phenomena is singular, so that the unperturbed system fails to capture the behavior of the system for $0 < \varepsilon \ll 1$.

4 OBSERVATIONS AND CONCLUSIONS

Using a multiple scales analysis, we have analyzed the effects of shear deformation on the resulting dynamical behavior of a system of coupled ordinary differential equations, modeling the $SO(2)$ symmetric response of elastic rings. Both shear deformation and rotatory inertia are accounted for in the model, and we have concentrated on the existence and dynamical consequences of a nontrivial state, reminiscent of "wobbling" solutions found by Healey within a transversely isotropic Cosserat rod model [10].

We find that nontrivial relative equilibria exist when the angular momentum of the system is above some critical value. Positive rotatory inertia of the material planes about the dynamic axis, described by the small parameter $\varepsilon$, is essential in this analysis. The positivity of $\varepsilon$ has also been used by Caffisch and Maddocks to show the existence of solutions and the stability of equilibrium solutions which minimize potential energy in a model of a slender rod [4].

This nontrivial state becomes dynamically significant when the radial displacement of the ring exceeds a critical value which depends on the material parameters as well as the magnitude of the hydrostatic forcing. Although the ring is no longer in a relative equilibrium state, the shear deformation nonetheless affects the response of the ring which is studied through a multiple scales analysis. We identify the lowest order behavior to this system and due to a bifurcation in the fast time-scale, the resulting simplified equations are non-analytic. However, our reduced system does capture the qualitative features of the motion. Such systems often arise in the context of mechanical systems with intermittent contacts [22], although, as we have shown, systems with motion on two fundamentally different time scales can also produce piecewise continuous vector fields.

In the above analysis we have concentrated only on the existence of a class of solutions which are $SO(2)$ symmetric. We have not, however, discussed the stability of these solutions under perturbations which do not respect $SO(2)$ symmetry. Rather, our stability results are based on the Routh reduced system of ordinary differential equations. Such discussions on the partial differential equations governing the ring may shed light on the numerical instabilities encountered by Dempsey and Gladwell [7]. There most certainly exist non-axisymmetric solutions which may bifurcate from the symmetric solution as parameters are varied. For example, Healey has examined the dynamical behavior of rotating loops, subject to general constitutive laws [9, 11]. In contrast to the rings considered here, Healey's loops have no bending stiffness, although he addresses both the question of existence and stability of solutions which do not possess $SO(2)$ symmetry. Our analysis, by virtue of its restriction to equations respecting $SO(2)$ symmetry, fails to address the stability of these solutions. We shall investigate this question further.

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