CONTROL OF TWO COUPLED PENDULAS WITH ZERO ENERGY CHANGE

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ABSTRACT
We study a system of two identical forced pendula, coupled with a linear torsional spring. Using the idea of energy transfer between the two oscillators, we derive a nonlinear control law to stabilize the in-phase motion. Trajectories of the controlled system are required to remain on the constant energy surface of the Hamiltonian.

1 INTRODUCTION
Conservation laws play an invaluable role in the dynamical behavior and analysis of mechanical systems and their mathematical models. In the presence of these conserved quantities, the evolution of a given dynamical system is constrained to lie on a lower dimensional surface in phase space defined by the conservation law, i.e., an invariant manifold. However, if the dynamical behavior of the system is undesirable, we may attempt to control the system through the addition of terms to the mathematical model. Unfortunately, these control terms often do not respect the conservation laws of the original system.

In general, conservative mechanical systems, i.e., those whose dynamical behavior is described by Hamilton's canonical equations [3], maintain their symplectic structure under the presence of external control [10]. However, under the action of the control, the Hamiltonian no longer represents the mechanical energy of the system and is, in general, not conserved. It is also possible that the controlled system satisfies a different conservation law than that of the uncontrolled system.

In many applications, the dynamical behavior of the controlled system satisfies one or more conservation laws, often due to symmetries inherited from the physical system. For example, there exists an extensive literature concerning the stability and stabilization of rotating structures, subject to the conservation of angular momentum. Simo, Posbergh, and Marsden [14] use the energy-momentum method [13] to discuss the dynamical stability of the relative equilibria of nonlinearly elastic bodies. Posbergh and Zhao [11] also use this method to stabilize uniform rotation of a rigid body about an arbitrary axis fixed in the body.

This work is concerned with the general question: Using the appropriate control terms, can we stabilize a specific trajectory of a given system, subject to conservation laws of the uncontrolled system? Stated in an alternative way, we wish to devise a control strategy that realizes the goal of stabilizing a particular trajectory, but does not effect conserved quantities of the system. The question is related to the zero dynamics of the system [5, 10].

2 EQUATIONS OF MOTION
We study a system of two identical forced pendula, coupled with a linear torsional spring under the influence of applied control. The nondimensionalized equations of motion are:

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1, \\
\dot{\omega}_1 &= T - \sin \theta_1 + \varepsilon (\theta_2 - \theta_1) + \eta f_1(t), \\
\dot{\theta}_2 &= \omega_2, \\
\dot{\omega}_2 &= T - \sin \theta_2 + \varepsilon (\theta_1 - \theta_2) + \eta f_2(t).
\end{align*}
\]
The parameter $\varepsilon$ is the (nondimensionalized) spring constant and $T$ is the applied torque. The terms dependent on $\eta$ represent the applied control. Note that for $\varepsilon = 0$ and $\eta = 0$, the two pendula decouple.

Each individual oscillator possesses two qualitatively different types of motion, oscillating solutions and whirling (or running) solutions [7, 15]. The phase portrait of either (uncoupled, uncontrolled) oscillator is shown in Figure 1. A homoclinic orbit connecting the unstable equilibrium point to itself separates the bounded, periodic motions from the unbounded, aperiodic solutions. The whirling solutions, which lie outside the homoclinic orbit, are characterized by unbounded growth in the angular position $\theta$. Note that we do not assume small torque. Systems of coupled pendula have been used to model coupled Josephson junctions [1, 2, 16] and can also be derived from the discretization of the undamped sine-Gordon equation [6].

### 2.1 Uncontrolled Dynamics

For nonzero coupling, the in-phase motion is neutrally stable for sufficiently low energy. However, trajectories which are initially in the neighborhood of the in-phase motion do not asymptotically approach $\theta_1 = \theta_2$ as $t \to \infty$. Instead they simply remain in a neighborhood of the in-phase motion for all time. A numerical example of this behavior is shown in Figure 2(a).

If the energy of the system is greater than some critical value, the periodic, in-phase motion is unstable. In fact, solutions which are initially oscillatory, can, for suitable energy levels, escape the region defined by the separatrix and, as a result, grow unbounded, (as shown in Figure 2(b)). In words, there is a transfer of energy between the two oscillators. One oscillator transfers energy to the second, allowing it to escape. Once the second oscillator is outside the separatrix, it transfers energy back to the first oscillator, “pulling” it out of its potential well. Throughout this process, the total energy of the system remains constant. A similar mechanism was described in Quinn [12].

### 2.2 Hamiltonian Structure

The uncontrolled system ($\eta = 0$) is sympletic, possessing the Hamiltonian $H(\theta_1, \omega_1, \theta_2, \omega_2; \varepsilon)$:

\[
H = \frac{\omega_1^2 + \omega_2^2}{2} - (\cos \theta_1 + \cos \theta_2) - \frac{T(\theta_1 + \theta_2)}{2} + \varepsilon \frac{(\theta_1 - \theta_2)^2}{2} = \hbar_0.
\]

As a result, Eq. (3), which can be viewed as the mechanical energy of the system, is conserved along orbits of the system.

Using this idea of energy transfer between two oscillators, we derive a nonlinear control law on $\tilde{f}_1$ so that the in-phase motion is stabilized. Specifically, we construct variational equations around $\theta_1 = \theta_2$ to identify the lowest order terms contributing to the instability. We then develop a suitable controller that is constrained to lie on the constant energy surface of the Hamiltonian, while making the in-phase motion asymptotically stable. As a result, stability is obtained by the controlled transfer of energy from one oscillator to the other.

### 3 Energy Conserving Control

Because of the sympletic structure of the uncontrolled system, Eq. (3) is conserved along trajectories. Unfortunately, the addition of the general control terms $f_1(t)$ and $f_2(t)$ destroy the invariance of this equation. However, provided the control satisfies an additional constraint, the Hamiltonian remains constant along trajectories of the controlled system. To accomplish this, we require $\dot{H}(t) = 0$:

\[
0 = \dot{H}(t), \quad \dot{H}(t) = \omega_1 \dot{\omega}_1 + \omega_2 \dot{\omega}_2 + \{\sin \theta_1 - T + \varepsilon(\theta_1 - \theta_2)\} \dot{\theta}_1 + \{\sin \theta_2 - T + \varepsilon(\theta_2 - \theta_1)\} \dot{\theta}_2, \quad \eta(\omega_1 f_1(t) + \omega_2 f_2(t)).
\]

**FIGURE 1**: Phase portrait of each individual oscillator (mod $2\pi$). The stable and unstable manifolds of the saddle point are highlighted. Note that the homoclinic orbit, which connects the saddle point to itself, separates bounded, periodic orbits (librations) from aperiodic trajectories which grow unbounded in forward time (rotations).

**FIGURE 2**: Uncontrolled dynamics; $\varepsilon = 0.1$, $T = 0.5$. The system is started from rest with given initial conditions ($\theta_2(t)$ is dotted).
Clearly, if we choose \( f_1 \) and \( f_2 \) such that \( \omega_1 f_1 + \omega_2 f_2 = 0 \), the system, with the additional control terms, still conserves Eq. (3). Although \( H \) no longer serves as a Hamiltonian for the controlled system, it nonetheless remains constant along solution curves.

4 CONTROLLER
The remaining task is to actually find a suitable controller which, subject to Eq. (4), stabilizes the in-phase motion. Consider the energy associated with each oscillator, given as:

\[
\mathcal{H}_i(\theta_i, \omega_i) = T_i(\omega_i) + V_i(\theta_i),
\]

\[
= \frac{\omega^2_i}{2} - \cos \theta_i - T \theta_i,
\]

where \( T_i \) and \( V_i \) are, respectively, the kinetic and potential energies of each oscillator. The total energy of the system, which, recall, is constant, is found to be:

\[
\mathcal{H} = \mathcal{H}_1(t) + \mathcal{H}_2(t) + \frac{\varepsilon (\theta_1 - \theta_2)^2}{2}.
\]

Thus, for trajectories which approach an in-phase motion as \( t \to \infty \), we would like \( \mathcal{H} \to \hbar_0/2 \). Taking the derivative of \( \mathcal{H}_i \) with respect to time, we find that, for \( i, j = 1, 2 \):

\[
\dot{\mathcal{H}}_i = \varepsilon \omega_i (\theta_j - \theta_i) + \omega_i f_i, \quad i \neq j.
\]  

We must choose \( f_i \) so that (i) \( \dot{\mathcal{H}}_i \to \hbar_0/2 \) and (ii) Eq. (4) is satisfied. Clearly, \( f_i \) must possess the following properties:

1. Because the in-phase motion, \( \theta_1 = \theta_2 = \omega_1 = \omega_2 \), is an exact solution to the uncontrolled system, the applied control must vanish when the motion is in-phase.

2. If \( f_i \) satisfies Eq. (4), we may write \( f_1 = \omega_2 f(t) \) and \( f_2 = -\omega_1 f(t) \).

From Eq. (5), we posit a control law based on the difference between the kinetic energy of the two oscillators, i.e., \( F(\mathcal{T}_1 - \mathcal{T}_2) \). Note however, it is the term \( \omega_i f_i \) that imparts control on the energy of each oscillator. Unfortunately, we cannot simply let, say, \( \omega_1 f_1 = \mathcal{T}_2 - \mathcal{T}_1 \). With such a choice, \( f_1 \) would be undefined whenever \( \omega_1 = 0 \).

To overcome this difficulty, we write:

\[
\omega_1 f_1 = \omega_1 \omega_2 f(t),
\]

\[
\omega_2 f_2 = -\omega_1 \omega_2 f(t).
\]

We must choose \( f(t) \) appropriately so that the in-phase motion is stabilized by \( \omega_i f_i \). Because \( \omega_i \) can take both positive and negative values, a simple linear feedback law might not be effective. Instead, let \( f(t) \) be of the form:

\[
f(t) = 2(\mathcal{T}_2 - \mathcal{T}_1) \text{sgn}(\omega_1 \omega_2).
\]

The resulting nonlinear control terms are found to be:

\[
f_1(t) = (\omega_2^2 - \omega_1^2) \text{sgn}(\omega_1)|\omega_2|, \quad (6a)
\]

\[
f_2(t) = (\omega_1^2 - \omega_2^2) \text{sgn}(\omega_2)|\omega_1|, \quad (6b)
\]

where \( \text{sgn}(\cdot) = \pm 1 \), and represents the sign of the argument. We assume that \( \text{sgn}(0) = 1 \).

\[\text{FIGURE 3: Numerical integration of Eqs. (7), with the initial conditions } \theta_1(0) = 2.1, \omega_1(0) = 0.0, \theta_2(0) = 1.9, \omega_2(0) = 0.0; \varepsilon = 0.1, T = 0.5. \text{ For } \eta = 0.10, \text{ the response remains bounded but is attracted to an out-of-phase motion.} \]

5 RESULTS
With the choice of control in Eq. (6), the equations of motion for the two coupled pendula become:

\[
\dot{\theta}_1 = \omega_1, \quad (7a)
\]

\[
\dot{\omega}_1 = T - \sin \theta_1 + \varepsilon(\theta_2 - \theta_1) + \eta (\omega_2^2 - \omega_1^2) \text{sgn}(\omega_1)|\omega_2|, \quad (7b)
\]

\[
\dot{\theta}_2 = \omega_2, \quad (7c)
\]

\[
\dot{\omega}_2 = T - \sin \theta_2 + \varepsilon(\theta_1 - \theta_2) + \eta (\omega_2^2 - \omega_1^2) \text{sgn}(\omega_2)|\omega_1|. \quad (7d)
\]

We illustrate the effect of this controller in Figure 3. With identical initial conditions, the in-phase motion of the uncontrolled system (\( \eta = 0 \)) is not only unstable, but initial conditions in the neighborhood of this state eventually escape the region of oscillations and begin whirling. For small \( \eta \), we find that the controlled in-phase motion is still unstable. However, for \( \eta = 0.1 \), the trajectory is attracted to an out-of-phase motion. Increasing the control further, to \( \eta = 0.25 \), we stabilize the in-phase motion.

In Figure 4 we present the region of stabilization for the in-phase motion of Eqs. (1). In each plot, initial conditions of the first oscillator are chosen randomly, while the initial state of the second oscillator is chosen so that the system lies on a constant energy surface (a similar procedure is used to generate initial conditions for Poincaré surfaces of multi-degree-of-freedom Hamiltonian systems). Marked initial conditions (in \( \theta_1 - \omega_1 \) space) lead to a trajectory that, after 50 time units, is close to the in-phase motion, so that \( (\theta_1 - \theta_2)^2 + (\omega_1 - \omega_2)^2 < 10^{-6} \). We assume that these initial
state, we study the controlled system for variations around the state \( x_2 \equiv 0 \). Thus we expand each variable in \( e \), where \( e \) is a small \(( \ll 1 \) scaling parameter, around \( x_2 = 0 \), so that:

\[
\begin{align*}
x_1 &= \theta + e\xi_1 + O(e^2), \\
y_1 &= \omega + e\mu_1 + O(e^2), \\
x_2 &= e\xi_2 + O(e^2), \\
y_2 &= e\mu_2 + O(e^2).
\end{align*}
\]

Substituting these expansions into Eqs. (7), we find that to lowest order in \( e \):

\[
\begin{align*}
O(1) : \dot{\theta} &= \omega, \\
\dot{\omega} &= T - \sin \theta, \\
\end{align*}
\]

and so the first order variational equations around \( e = 0 \) (or equivalently \( x_2 = 0 \)) become:

\[
\begin{align*}
O(e) : \dot{\xi}_1 &= \mu_1, \\
\mu_1 &= - (\cos \theta) \xi_1, \\
\dot{\xi}_2 &= \mu_2, \\
\mu_2 &= - (\cos \theta + 2e) \xi_2 - 2\eta \omega^2 \mu_2.
\end{align*}
\]

The stability of the in-phase motion is determined by the stability of the zero solution of Eqs. (10). Note that the transformation given in Eq. (8) decouples the dynamics of \((\xi_1, \mu_1)\) and \((\xi_2, \mu_2)\) in the resulting first variational equations. Physically, deviations that remain in-phase do not effect the resulting behavior of deviations that are exactly out-of-phase.

To determine the stability of the trajectory \( \theta_1 = \theta_2 \equiv \theta \), we use Floquet theory, which we summarize here (see Nayfeh and Mook [9] or Meirovitch [8] for more details). In general, we assume the following \( n \)-dimensional system:

\[
\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n
\]

possesses a \( T \)-periodic solution, represented by \( \mathbf{X} \), i.e., \( \mathbf{X}(t) = \mathbf{X}(t + T) \). The stability of this solution is determined by the variational equations, which can be written as:

\[
\dot{\mathbf{\xi}} = \nabla \mathbf{f}(\mathbf{X}) \mathbf{\xi}.
\]

Note that by construction these equations are linear. Consequently, the solution to this system, \( \mathbf{\xi}(t) \), can be obtained from the fundamental solution matrix \( \Phi(t) \in \mathbb{R}^{n \times n} \) [4]:

\[
\mathbf{\xi}(t) = \Phi(t) \mathbf{\xi}(0),
\]

where \( \mathbf{\xi}(0) \) represent the initial conditions of this system. Letting \( \Xi(t) = [\xi_1(t) \cdots \xi_n(t)] \), we choose \( n \) linearly independent initial conditions of the form \( \Xi(0) = \text{Id}(n) \), the \( n \times n \) identity matrix, so that the evolution of \( \Xi(t) \) is determined by:

\[
\dot{\Xi} = \nabla \mathbf{f}(\mathbf{X}) \Xi, \quad \Xi(0) = \text{Id}(n).
\]
As a result, the solution to this set of initial conditions is simply:

\[ \Xi(t) = \Phi(t)\Xi(0), \]
\[ = \Phi(t). \] (12)

Because \( X \), the solution to the original system, is periodic in time with period \( T \), Eq. (12) induces a time-\( T \) return map of the form:

\[ \Xi_{k+1} = \Phi(T)\Xi_k, \quad \Xi_m = \Xi(mT). \]

Consequently, the stability of the zero solution to the variational equations is determined by the eigenvalues of the matrix \( \Phi(T) \), also known as the Floquet multipliers. Because Eqs. (7) are autonomous, one of the Floquet multipliers is simply +1, corresponding to the invariance of the system under shifts in time. If the modulus of the remaining Floquet multipliers is less than one, the \( T \)-periodic solution is asymptotically stable. However, if one or more of the Floquet multipliers has modulus greater than one, the solution is unstable. \( \Phi(T) \) is found by numerically integrating Eq. (11) for \( T \) units of time. Although numerical integration must still be used to determine stability, Floquet theory allows us to determine this crucial property of the solution using only a finite time integration with one set of initial conditions.

Eqs. (10a-10b), the variational equations governing deviations in-phase, form a Hill’s equation, so that the product of its Floquet multipliers is unity [8]. We notice that these equations possess a \( T \)-periodic solution of the form \( \xi_1 = \omega \), so that one of its Floquet multipliers is unity. Consequently, the second Floquet multiplier is also unity, and the amplitude of the second linearly independent solution grows linearly in time. This reflects the period-amplitude dependence of the in-phase mode which satisfies Eqs. (9). However, the in-phase motion is orbitally stable with respect to deviations that remain in-phase. So, the stability of the in-phase motion is determined by the response to perturbations in the \( \xi_2 - \mu_2 \) direction.

In Figure 5, we examine the stability of the in-phase motion as \( \eta \) is varied. In this figure, \( \theta_0 \) specifies the in-phase motion which possesses the initial conditions \( \theta_1(0) = \theta_2(0) \equiv \theta_0, \omega_1(0) = \omega_2(0) \equiv 0 \). \( \theta_0 \) is chosen to lie between the neutrally stable equilibrium point and the saddle point, at \( \theta_0 = \arcsin(T) \) and \( \theta_0 = \pi - \arcsin(T) \) respectively (with \( -\pi/2 < \arcsin(T) < \pi/2 \)). For this range of \( \theta_0 \), all in-phase motions are sampled, cf. Figure 1. In each plot, points in \( \theta_0 - \varepsilon \) space are randomly chosen, with a fixed value for \( \eta \). The eigenvalues of \( \Phi(T) \) are then found for each point. Marked points correspond to a linearly stable in-phase motion under the appropriate value of \( \eta \).

For \( \eta = 0 \), Eqs. (10c-10d) also form a Hill’s equation. Again, the product of the Floquet multipliers is unity, so that, at best, the in-phase motion is only neutrally stable. In fact, there exists a large region of instability for small \( \varepsilon \).

As \( \eta \) increases from zero, the region of instability decreases and we find that the in-phase motion becomes increasingly asymptotically stable to deviations in the \( \xi_2 - \mu_2 \) direction, i.e., out-of-phase perturbations. Recall that the in-phase motion is only orbitally stable to deviations in the \( \xi_1 - \mu_1 \) direction. Surprisingly, as seen in Figure 5d, even for \( \eta = 1.0 \) the in-phase motion remains unstable for small \( \varepsilon \) as \( \theta_0 \) approaches the saddle point (cf. Figure 1), so that the \( T \)-periodic in-phase solution approaches the homoclinic orbit (\( T \to \infty \)). It is suspected that this instability arises from the period-amplitude dependence of the in-phase motion.

7 CONCLUSIONS

In this work we develop a suitable controller which dramatically increases the region of stability for the in-phase motion of two coupled pendula. The control terms are chosen to not only stabilize \( \theta_1 \equiv \theta_2 \), but to conserve the mechanical energy of the original system as well. The region of stability of the in-phase motion is described both numerically and analytically, the latter via Floquet theory.

Linear derivative feedback control, which is often used to asymptotically stabilize a state, always decreases the energy of the system. As a result, out-of-phase perturbations, even those that leave the energy unchanged, are always followed by a reduction of the mechanical energy. In the presence of such a perturbation, although we might be able to
again bring the system in-phase, we no longer approach the same trajectory. Rather, we approach an in-phase state with lower energy. In contrast, the dynamical behavior of Eqs. (7) will approach the in-phase motion with the same energy as the initial conditions, provided the perturbation leaves the energy unchanged.

Future work will include extensions to $n$ coupled oscillators and the effects of non-identical components. Also, the general question concerning the dynamical behavior of a controlled system, subject to conservation laws of the uncontrolled system, is still unanswered. In such a situation the control vector field is required to be tangent to the invariant submanifold defined by the conservation law (cf., Eq. (4)). Presumably, once the controlled dynamics are restricted to this manifold, the stabilization of any invariant set within the submanifold (such as the in-phase motion in this example) will become a standard problem in nonlinear control theory.

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