CONTROL OF TWO COUPLED PENDULA WITH ZERO ENERGY CHANGE*

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We study a system of two identical forced pendula, coupled with a linear torsional spring. Using the idea of energy transfer between the two oscillators, we derive a nonlinear control law to stabilize the in-phase motion. Trajectories of the controlled system are required to remain on the constant energy surface of the Hamiltonian.

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Control in the Presence of Conservation Laws

- The evolution of the uncontrolled dynamical system is constrained to lie on a lower dimensional surface in phase space defined by the conservation law, i.e., an invariant manifold. The invariant surfaces often arise from symmetries in the physical system.

- Unfortunately, control terms often do not respect the conservation laws of the original system.

Can we stabilize a specific trajectory of a given system, subject to conservation laws of the uncontrolled system? Stated in an alternative way, we wish to devise a control strategy that realizes the goal of stabilizing a particular trajectory, but does not effect conserved quantities of the system.
**Dynamical System**

We study a system of two identical forced pendula, coupled with a linear torsional spring.

Under the influence of applied control, the nondimensional equations of motion are:

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1, \quad (1a) \\
\dot{\omega}_1 &= T - \sin \theta_1 + \varepsilon(\theta_2 - \theta_1) + \eta f_1(t), \quad (1b) \\
\dot{\theta}_2 &= \omega_2, \quad (1c) \\
\dot{\omega}_2 &= T - \sin \theta_2 + \varepsilon(\theta_1 - \theta_2) + \eta f_2(t). \quad (1d)
\end{align*}
\]

\(\varepsilon\) : nondimensionalized spring constant; \\
\[0 < \varepsilon \ll 1\]

\(T\) : nondimensionalized applied torque; \\
\[0 < T < 1\]
The terms dependent on $\eta$ represent the applied control. Note that for $\varepsilon = 0$ and $\eta = 0$, the pendula decouple and the system is integrable.
**Individual Oscillators**

Possess two qualitatively different types of motion: oscillating solutions and whirling (or running) solutions.

**Homoclinic orbit**: connects the unstable equilibrium point to itself and separates the bounded, periodic motions from the unbounded, aperiodic solutions.
The whirling solutions, which lie outside the homoclinic orbit, are characterized by unbounded growth in the angular position $\theta$. 
Coupled System

The uncontrolled system ($\eta = 0$) is symplectic, possessing the Hamiltonian $\mathcal{H}(\theta_1, \omega_1, \theta_2, \omega_2; \varepsilon)$:

$$\mathcal{H} = \frac{\omega_1^2 + \omega_2^2}{2} - (\cos \theta_1 + \cos \theta_2) - T(\theta_1 + \theta_2) + \varepsilon\frac{(\theta_1 - \theta_2)^2}{2},$$

$$= h_0.$$

As a result, the Hamiltonian, which is simply the mechanical energy of the system, is conserved along orbits of the system.

**High energies:** the in-phase motion is a whirling solution. We are attempting to stabilize a periodic motion.
(a) $\theta_1(0) = 1.10, \quad \theta_2(0) = 1.30$

(b) $\theta_1(0) = 2.01, \quad \theta_2(0) = 1.99$

**Low energies:** The in-phase motion, a periodic, small-amplitude motion, is neutrally stable. Trajectories do not asymptotically approach $\theta_1 = \theta_2$ as $t \to \infty$. Rather, they simply remain in a neighborhood of the in-phase motion for all time.

**Intermediate energies:** The in-phase motion exists near the homoclinic orbit, and is unstable. In fact, solutions which are initially oscillatory, can, for suitable energy levels, escape the oscillatory region and grow unbounded.
In words, there is a transfer of energy between the two oscillators. One oscillator transfers energy to the second, allowing it to escape. Once the second oscillator is outside the separatrix, it transfers energy back to the first oscillator, “pulling” it out of its potential well. Throughout this process, the total energy of the system remains constant.
Using this idea of energy transfer between two oscillators, we derive a \textit{nonlinear} control law on $f_i$ so that the in-phase motion is stabilized. Specifically, we:

- Develop a suitable controller that is constrained to lie on the constant energy surface of the mechanical energy, while making the in-phase motion asymptotically stable.

- Construct variational equations around the in-phase motion to identify the lowest order terms contributing to the instability and study the performance of the controller.

\textbf{Stability is obtained by the controlled transfer of energy from one oscillator to the other.}
Control Terms

The addition of general control terms $f_1(t)$ and $f_2(t)$ destroy the conservation of the mechanical energy. However, provided the control satisfies an additional constraint, the mechanical energy is again conserved.

We require $\dot{\mathcal{H}}(t) = 0$:

$$\dot{\mathcal{H}}(t) = \omega_1 \dot{\omega}_1 + \omega_2 \dot{\omega}_2 + \left\{ \sin \theta_1 - T + \varepsilon (\theta_1 - \theta_2) \right\} \dot{\theta}_1 + \left\{ \sin \theta_2 - T + \varepsilon (\theta_2 - \theta_1) \right\} \dot{\theta}_2,$$

$$= \eta (\omega_1 f_1(t) + \omega_2 f_2(t)).$$

If we choose $f_1$ and $f_2$ such that $\omega_1 f_1 + \omega_2 f_2 = 0$, the system, with the additional control terms, still conserves the mechanical energy of the system.
The mechanical energy of the uncontrolled system is conserved along trajectories.

Although $\mathcal{H}$ no longer serves as a Hamiltonian for the controlled system, it nonetheless remains constant along solution curves.
Energy Based Control

Consider the energy associated with each oscillator, given as:

\[ \mathcal{H}_i(\theta_i, \omega_i) = \mathcal{T}_i(\omega_i) + \mathcal{V}_i(\theta_i), \]

\[ = \frac{\omega_i^2}{2} - \cos \theta_i - T\theta_i, \]

\( \mathcal{T}_i \) : Kinetic energy
\( \mathcal{V}_i \) : Potential energy

The total mechanical energy can be written:

\[ \mathcal{H} = \mathcal{H}_1(t) + \mathcal{H}_2(t) + \varepsilon \frac{(\theta_1 - \theta_2)^2}{2}, \]

\[ = h_0. \]

For trajectories which approach an in-phase motion as \( t \to \infty \), we see that \( \mathcal{H}_i \to h_0/2 \). Taking the derivative of \( \mathcal{H}_i \) with respect to time, we find that, for \( i, j = 1, 2 \):

\[ \dot{\mathcal{H}}_i = \varepsilon \omega_i (\theta_j - \theta_i) + \omega_i f_i, \quad j \neq i. \]
Requirements and Properties

To stabilize the in-phase motion, we require:

- \( \mathcal{H}_i \to h_0/2 \) as \( t \to \infty \),
- \( \omega_1 f_1(t) + \omega_2 f_2(t) = 0 \).

Also, the control terms must possess the properties:

1. \( f_1 \) and \( f_2 \) must vanish when \( \theta_1 \equiv \theta_2 \),
2. We may consider:
   \[
   f_1(t) = \omega_2 f(t), \quad f_2(t) = -\omega_1 f(t).
   \]

We posit a control law based on the difference between the kinetic energy of the two oscillators, i.e., \( F(T_1 - T_2) \). Note however, it is the term \( \omega_i f_i \) that imparts control on the energy of each oscillator:

\[
\begin{align*}
\omega_1 f_1 &= \omega_1 \omega_2 f(t), \\
\omega_2 f_2 &= -\omega_1 \omega_2 f(t).
\end{align*}
\]
Unfortunately, we cannot simply let, say, $\omega_1 f_1 = \mathcal{T}_2 - \mathcal{T}_1$. With such a choice, $f_1$ would be undefined whenever $\omega_1 = 0$.

1. Because the in-phase motion, $\theta_1 = \theta_2$, $\omega_1 = \omega_2$, is an exact solution to the uncontrolled system, the applied control must vanish when the motion is in-phase.

2. We may write $f_1 = \omega_2 f(t)$ and $f_2 = -\omega_1 f(t)$. 
We must choose \( f(t) \) appropriately so that the in-phase motion is stabilized by \( \omega_i f_i \). We let \( f(t) \) be of the form:

\[
f(t) = 2(T_2 - T_1) \text{sgn}(\omega_1 \omega_2).
\]

The resulting nonlinear control terms are found to be:

\[
f_1(t) = (\omega_2^2 - \omega_1^2) \text{sgn}(\omega_1)|\omega_2|,
\]

\[
f_2(t) = (\omega_1^2 - \omega_2^2) \text{sgn}(\omega_2)|\omega_1|,
\]

where \( \text{sgn}(\cdot) = \pm 1 \), and represents the sign of the argument. With this choice for the control terms, the equations of motion become:

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1, \quad \text{(2a)} \\
\dot{\omega}_1 &= T - \sin \theta_1 + \varepsilon(\theta_2 - \theta_1) + \eta(\omega_2^2 - \omega_1^2) \text{sgn}(\omega_1)|\omega_2|, \quad \text{(2b)} \\
\dot{\theta}_2 &= \omega_2, \quad \text{(2c)} \\
\dot{\omega}_2 &= T - \sin \theta_2 + \varepsilon(\theta_1 - \theta_2) + \eta(\omega_1^2 - \omega_2^2) \text{sgn}(\omega_2)|\omega_1|. \quad \text{(2d)}
\end{align*}
\]
Because $\omega_i$ can take both positive and negative values, a simple linear feedback law might not be effective.
Response of the Controlled System

\[ \varepsilon = 0.1, \quad T = 0.5 \]

\[
\begin{align*}
\theta_1(0) &= 2.1, & \omega_1(0) &= 0.0, \\
\theta_2(0) &= 1.9, & \omega_2(0) &= 0.0,
\end{align*}
\]

The in-phase motion is asymptotically stable for \( \eta \) sufficiently large.
With identical initial conditions, the in-phase motion of the uncontrolled system ($\eta = 0$) is not only unstable, but initial conditions in the neighborhood of this state eventually escape the region of oscillations and begin whirling. For small $\eta$, we find that the controlled in-phase motion is still unstable. However, for $\eta = 0.1$, the trajectory is attracted to an out-of-phase motion. Increasing the control further, to $\eta = 0.25$, we stabilize the in-phase motion.
Basin of Attraction—Constant Energy

(a) $h_0 = -2.08$,  
$\eta = 1.00$

(b) $h_0 = -1.64$,  
$\eta = 1.00$

(c) $h_0 = -1.17$,  
$\eta = 0.50$

(d) $h_0 = -1.17$,  
$\eta = 1.00$

Marked initial conditions (in $\theta_1 - \omega_1$ space) lead to a trajectory that asymptotically approaches an in-phase motion as $t \to \infty$.  

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In each plot, initial conditions of the first oscillator are chosen randomly, while the initial state of the second oscillator is chosen so that the system lies on a constant energy surface. Marked initial conditions (in $\theta_1 - \omega_1$ space) lead to a trajectory that, after 50 time units, is close to the in-phase motion, so that:

$$(\theta_1 - \theta_2)^2 + (\omega_1 - \omega_2)^2 < 10^{-6}.$$ 

We assume that these initial conditions asymptotically approach an in-phase motion as $t \to \infty$.

In particular, we note that initial conditions which are close to the in-phase motion are stabilized. This can be compared to the uncontrolled system, in which the in-phase motion is, at best neutrally stable, and often unstable. The fine structure of this region of stabilization away from the in-phase motion is, at present, unexplained, however, we suspect that it stems from the nonanalytic form of the control law.
**Stability of the In-phase state**

We wish to investigate the effects of \( \eta \) on the ability of the control terms to stabilize the in-phase motion. To simplify Eqs. (2), we make the following transformation:

\[
\begin{align*}
x_1 &= \frac{\theta_1 + \theta_2}{2}, & x_2 &= \frac{\theta_1 - \theta_2}{2}, \\
y_1 &= \frac{\omega_1 + \omega_2}{2}, & y_2 &= \frac{\omega_1 - \omega_2}{2}.
\end{align*}
\]  

If the in-phase motion is stable, \( x_2 \to 0 \) as \( t \to \infty \). To determine the stability of this state, we study the controlled system for variations around the state \( x_2 \equiv 0 \) via Floquet theory.

We expand each variable in \( e \), where \( e \) is a small (\( \ll 1 \)) scaling parameter, around \( x_2 = 0 \), so that:

\[
\begin{align*}
x_1 &= \theta + e\xi_1 + \mathcal{O}(e^2), \\
y_1 &= \omega + e\mu_1 + \mathcal{O}(e^2), \\
x_2 &= e\xi_2 + \mathcal{O}(e^2), \\
y_2 &= e\mu_2 + \mathcal{O}(e^2).
\end{align*}
\]
Recall that $\eta$ characterizes the strength of the control.

$x_2$ and $y_2$ measure the deviation of a trajectory from the in-phase state.
Substituting these expansions into Eqs. (2), we find that to lowest order in \( e \):

\[
\mathcal{O}(1): \dot{\theta} = \omega, \\
\dot{\omega} = T - \sin \theta,
\]

and the first order, linear variational equations around \( e = 0 \) (or equivalently \( x_2 = 0 \)) become:

\[
\mathcal{O}(e): \dot{\xi}_1 = \mu_1, \quad (4a) \\
\dot{\mu}_1 = - (\cos \theta) \xi_1, \quad (4b) \\
\dot{\xi}_2 = \mu_2, \quad (4c) \\
\dot{\mu}_2 = - (\cos \theta + 2e) \xi_2 - (2\eta \omega^2) \mu_2. \quad (4d)
\]

The stability of the in-phase motion is determined by the stability of the zero solution of Eqs. (4), which we determine through Floquet theory.
The $\mathcal{O}(1)$ system simply describes the dynamical behavior of the in-phase motion.

Note that the transformation to $(x_1, y_1, x_2, y_2)$ decouples the dynamics in the resulting first variational equations. Physically, deviations that remain in-phase do not effect the resulting behavior of deviations that are exactly out-of-phase.

Note that the equations on $\xi_1$ and $\mu_1$ form a Hill’s equation. In addition, since they possess one periodic solution of the form $\xi_1 = \omega$, this system is neutrally stable. Physically, in the in-phase motion is orbitally stable with respect to perturbations that preserve the in-phase motion.
Stability of the in-phase motion

(a) $\eta = 0$  
(b) $\eta = 0.25$

(c) $\eta = 0.50$  
(d) $\eta = 1.00$

Marked points correspond to a linearly stable in-phase motion under the appropriate value of $\eta$. 

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Floquet theory allows us to determine the stability of a periodic motion. Essentially, we reduce the flow in the neighborhood of the periodic orbit to a time $T$ return map. The Floquet multipliers then correspond to the eigenvalues of this return map.

$\theta_0$ specifies the in-phase motion which possesses the initial conditions $\theta_1(0) = \theta_2(0) \equiv \theta_0$, $\omega_1(0) = \omega_2(0) \equiv 0$. In each plot, points in $\theta_0-\varepsilon$ space are randomly chosen, with a fixed value for $\eta$. The Floquet multipliers are then found. Stable motion possess all multipliers within the unit circle.
Recall, the uncontrolled system is Hamiltonian, so that the in-phase motion is, at best, only neutrally stable. In fact, there exists a large region of instability for small $\varepsilon$.

For $\eta = 0$, the in-phase motion is only neutrally stable. The product of the Floquet multipliers is unity, so that, at best, the in-phase motion is only neutrally stable. In fact, there exists a large region of instability for small $\varepsilon$.

As $\eta$ increases from zero, the region of instability decreases and we find that the in-phase motion becomes asymptotically stable to deviations in the $\xi_2 - \mu_2$ direction, i.e., out-of-phase perturbations. Recall that the in-phase motion is only orbitally stable to deviations in the $\xi_1 - \mu_1$ direction, i.e., in-phase perturbations.

Surprisingly, even for $\eta = 1.0$ the in-phase motion remains unstable for small $\varepsilon$ as the periodic motion approaches the homoclinic orbit. We speculate that this instability results from the strong period-amplitude dependence of each uncoupled oscillator near the homoclinic orbit.
For $\eta = 0$, the variational equations form a Hill’s equation. The product of the Floquet multipliers is unity, so that, at best, the in-phase motion is only neutrally stable.
Conclusions

In this work we develop a suitable controller which dramatically increases the region of stability for the in-phase motion of two coupled pendula. The control terms are chosen to:

- **Stabilize the in-phase motion**, $\theta_1 \equiv \theta_2$,

- **Conserve the mechanical energy** of the original system.

Linear derivative control decreases the energy of the system. In the presence of an out-of-phase perturbation, although we might be able to again bring the system in-phase, we no longer approach the *same* trajectory without some reference input. Rather, we approach an in-phase state with lower energy. In contrast, the dynamical behavior of our controlled system will approach the in-phase motion with the same energy as the initial conditions, provided the perturbation leaves the energy unchanged.
Linear derivative feedback control, which is often used to asymptotically stabilize a state, always decreases the energy of the system. As a result, out-of-phase perturbations, even those that leave the energy unchanged, are always followed by a reduction of the mechanical energy. In the presence of such a perturbation, although we might be able to again bring the system in-phase, we no longer approach the same trajectory. Rather, we approach an in-phase state with lower energy. In contrast, the dynamical behavior of Eqs. (2) will approach the in-phase motion with the same energy as the initial conditions, provided the perturbation leaves the energy unchanged.